Cloud 9 Tutorials

F.A. Cup Draw Tutorial

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This tutorial is to help students to make a start with the F.A. Cup Draw worksheet.

To begin with, suppose that two of the four teams which make it into the semi-finals of the F.A. Cup are Premier League teams.

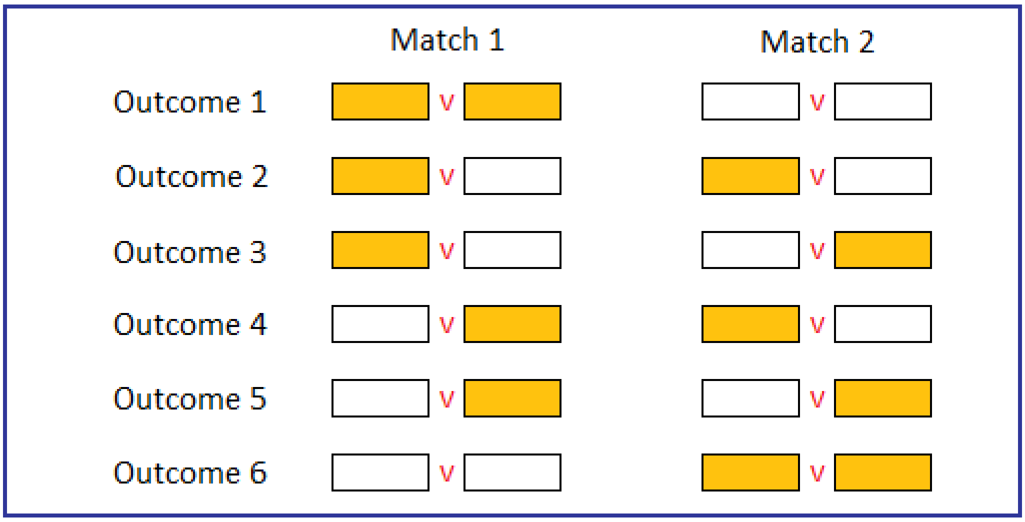
What is the probability that the two Premier League teams will avoid each other in the draw for the semi-finals?

Consider the structure of the draw.

Indicate a Premier League team as follows:

(Leave blanks for non-Premier League teams.)

Show the different ways that the two Premier League teams can be drawn.



Outcome 1 and Outcome 6 each involve 1 all-Premier League match. So in all the other arrangements there are no all-Premier League matches.

Since four of the six possible ways to draw the teams involve the Premier League teams avoiding each other then:

the probability that the Premier League teams avoid each other $= \frac{4}{6} \equiv \frac{2}{3}$

From a logical point of view, a Premier League team faces three possible opponents of which one is also a Premier League team.

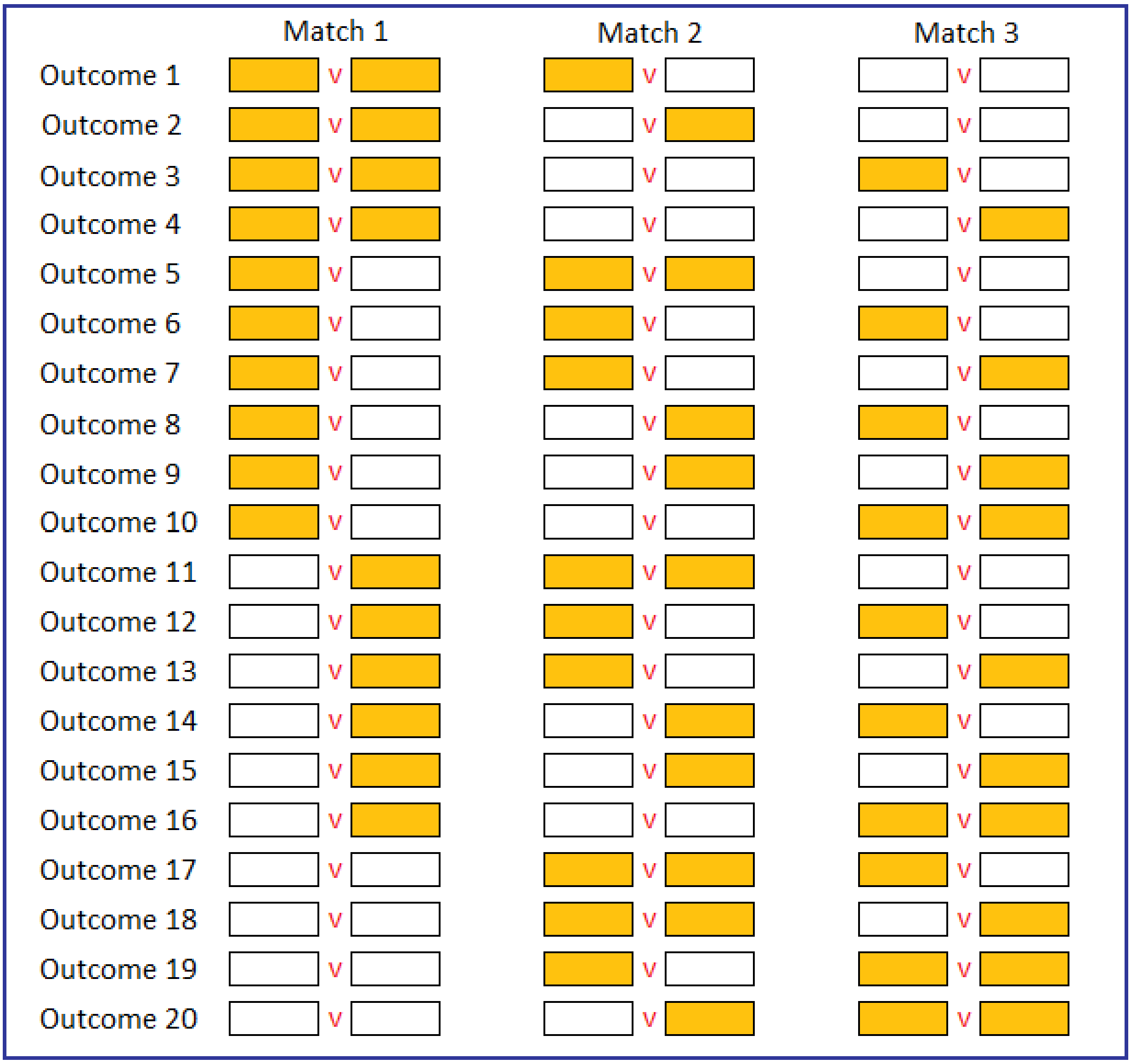
So the probability that the Premier League teams avoid each other must be $\frac{2}{3}$

This confirms the result using the arrangements method.

Next, although the F.A. Cup does not involve a round with three matches of six teams, mathematically it is worth considering this case as it can help build up patterns in the results.

What is the probability that the three Premier League teams will avoid each other in a draw for a round with three matches?

Show the different ways that the three Premier League teams can be drawn.

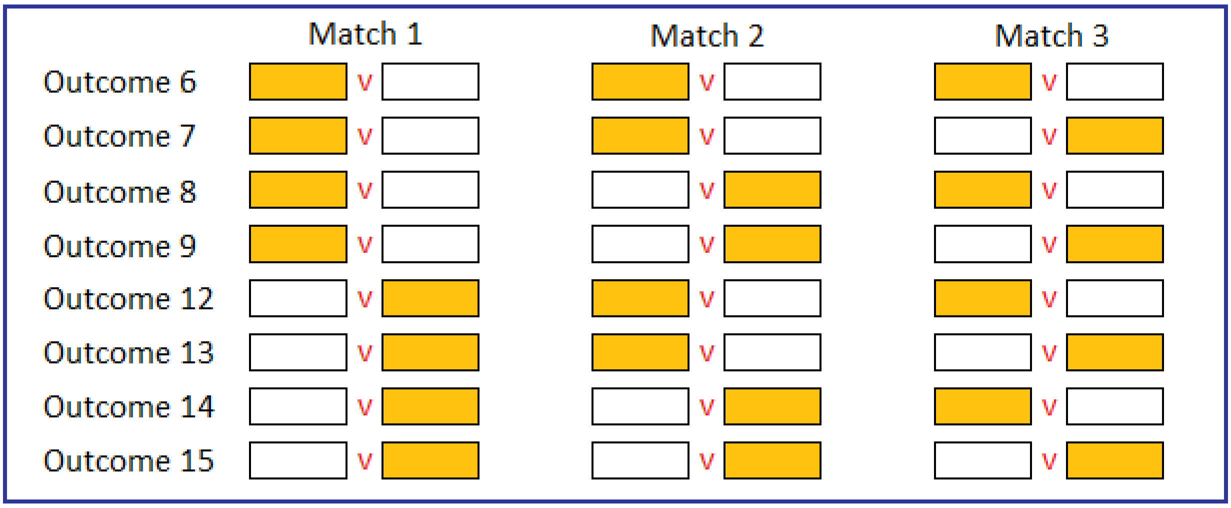


Examine the possible outcomes in which the Premier League teams avoid each other.

The outcomes shown above in which the Premier League teams avoid each other are:

Outcome 6	Outcome 12
Outcome 7	Outcome 13
Outcome 8	Outcome 14
Outcome 9	Outcome 15

Recap: There are eight outcomes in which the Premier League teams avoid each other.



Arrangements

Note that there are two positions for a Premier League team in Match 1, two positions for a Premier League team in Match 2 and two positions for a Premier League team in Match 3.

The total number of arrangements	=	Number of positions for a Premier League team in Match 1	x	Number of positions for a Premier League team in Match 2	x	Number of positions for a Premier League team in Match 3
	=	2	x	2	x	2
	=	2^3				
	=	8				

[Note: This is an extension of the "RS principle".

If there are R ways to do one thing and S ways to do another thing then the number of ways to do both things is R x S.

So if there are R x S ways to do two things and T ways to do another thing , the number of ways to do all three things is R x S x T.]

The probability of the three Premier League teams avoiding each other

$$= \frac{\text{the number of outcomes with no Premier League matches}}{\text{the total number of outcomes}} = \frac{8}{20} \equiv \frac{2}{5}$$

Arranging Items in a Row

Factorials

Note, 0! ("zero factorial") is defined as 1.

1! = 1

2! = 2 x 1 = 2

3! = 3 x 2 x 1 = 6

4! = 4 x 3 x 2 x 1 = 24

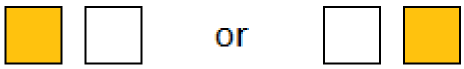
5! = 5 x 4 x 3 x 2 x 1 = 120

6! = 6 x 5 x 4 x 3 x 2 x 1 = 720

.... etc.

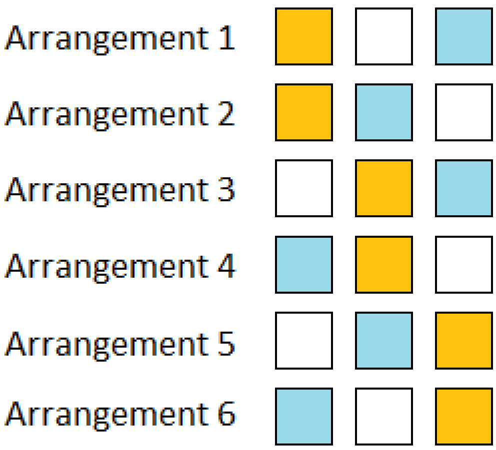
Consider two different-coloured squares.

There are two ways to arrange these squares in a row.



So the answer is given by 2!.

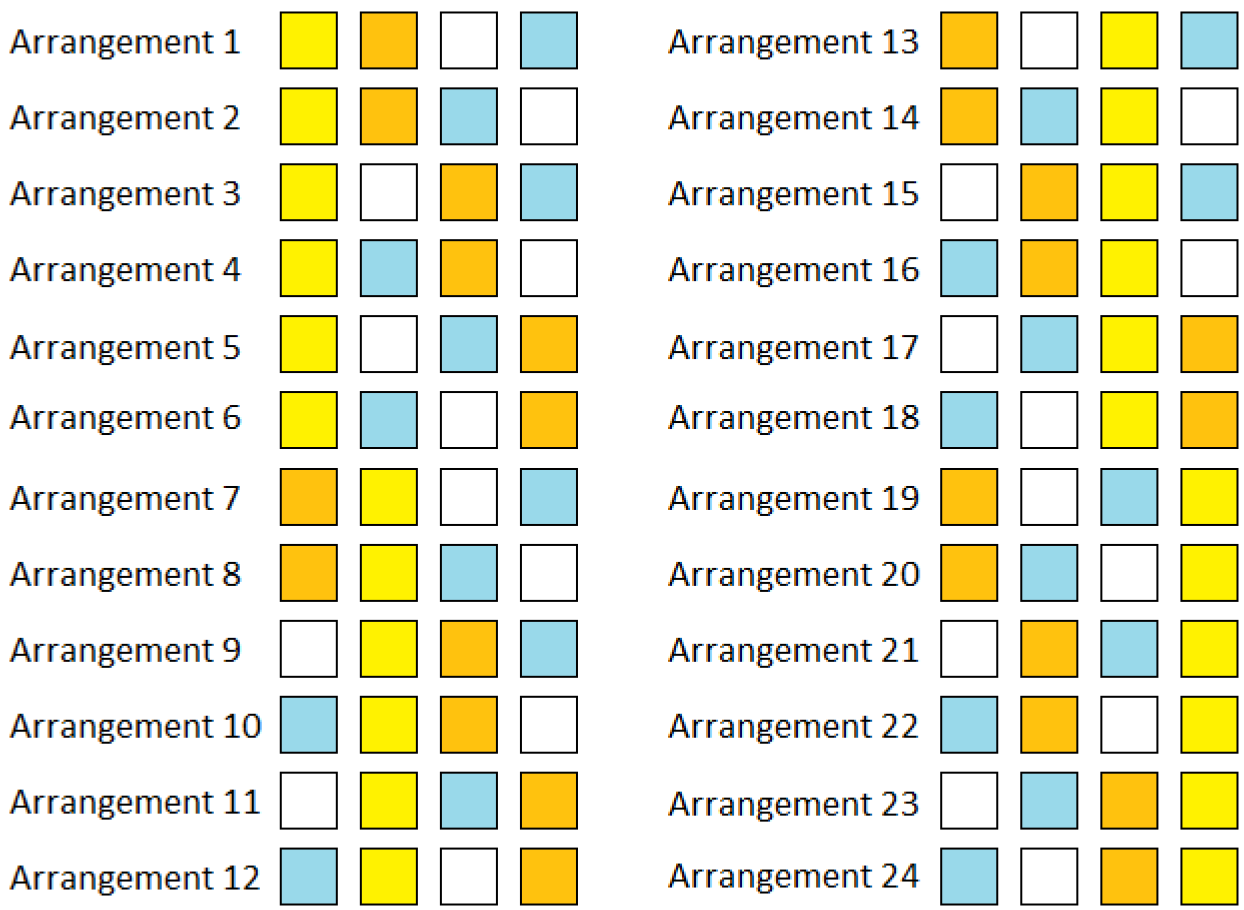
Now consider the number of ways to arrange 3 different-coloured squares in a row.



So there are 6 ways to arrange 3 different-coloured squares in a row.

Therefore, the answer is given by 3!

Next, with four different-coloured squares in a row, predict that the number of arrangements is 4!.



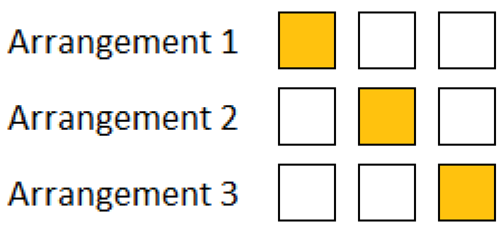
So there are 24 ways to arrange the four squares in a row.

Therefore, the answer is given by 4!

It can be deduced that the number of ways to arrange n different-coloured squares in a row = n!

Now consider the case where some of the squares have the same colour.

Find the number of ways to arrange three squares in a row where two of them have the same colour.



So there are 3 ways to arrange these squares.

This answer can be worked out using factorials as follows.

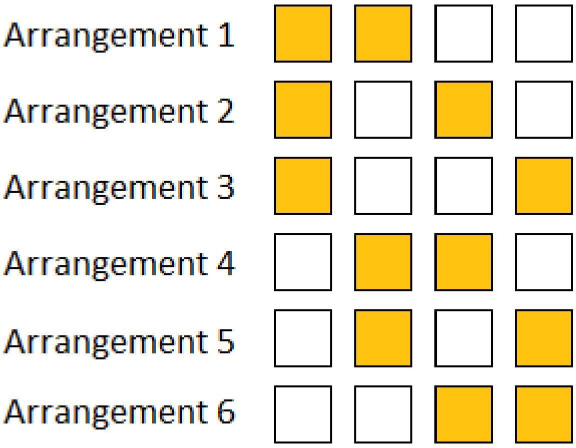
3 squares in a row.

$$\frac{3!}{1!\,2!} = 3$$

1 orange square

2 white squares

Find the number of ways to arrange four squares in a row where two of them have one colour and the other two squares have another colour.



So there are 6 ways to arrange these squares.

This answer can be worked out using factorials as follows.

squares in a row.

$$\frac{4!}{2!\,2!} = 6$$

2 orange squares   2 white squares

Note that this situation is the same as that used to find the number of outcomes when drawing two Premier League teams from four teams in the semi-final draw.

In general, the number of ways to arrange n squares in a row where r have one colour and (n – r) have another colour is given by:

$$\frac{n!}{r!\,(n-r)!}$$

This is known as the binomial coefficient which is abbreviated as: ${}^nC_r$

Example: Earlier the number of outcomes when drawing three Premier League teams from six teams was considered.

This result can be found by substituting n = 6 and r = 3 in the above formula.

$${}^6C_3 = \frac{6!}{3!\,3!} \equiv \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} \equiv \frac{\cancel{6} \times 5 \times 4}{\cancel{6}} \equiv 20 \quad \text{(as previously shown).}$$

Probability Distribution Table

Now produce a probability distribution table for a three-match draw.

Let the number of Premier League teams in the draw = t.

Let the number of all-Premier League matches = m.

	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 0				
t = 1				
t = 2				
t = 3				
t = 4				
t = 5				
t = 6				

Fill in the easy cases first.

If there are no Premier League team in the draw then it is certain that there will be no all-Premier League matches.

The first row (t = 0) can be filled-in as follows.

	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 0	1	0	0	0

If there is 1 only Premier League team in the draw then it is certain that there will be no all-Premier League matches.

So the t = 1 row can be filled-in as follows.

	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 1	1	0	0	0

If there are five Premier League teams in the draw then it is certain that there will be 2 all-Premier League matches.

So the t = 5 row can be filled-in as follows.

	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 5	0	0	1	0

If there are six Premier League teams in the draw then it is certain that there will be 3 all-Premier League matches.

So the t = 6 row can be filled-in as follows.

	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 6	0	0	0	1

Recap:

	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 0	1	0	0	0
t = 1	1	0	0	0
t = 2				
t = 3				
t = 4				
t = 5	0	0	1	0
t = 6	0	0	0	1

Find the probability distribution when **three** of the six teams are Premier League teams.

The total number of possible outcomes = ${}^{6}C_{3}$ (= 20)

[Keep the formula notation to detect patterns in the results.]

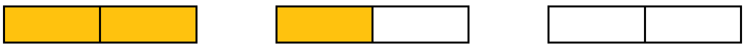
It was shown that the number of ways to pick out the six teams so that all three Premier League teams avoid each other = 2^3 (= 8).

So the probability that the three Premier League teams avoid each other $\equiv$ P (m = 0) = $\dfrac{2^3}{{}^{6}C_{3}}$

All the other possibilities each involve one all-Premier League match.
These 12 outcomes can be identified from the list of outcomes on page 2.

However, a more succinct method to find these 12 outcomes can be used.

Set two of the Premier League teams together.



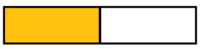
Treat these as three "double rods".

The number of ways to arrange the "Premier League double rod" in a row of three double rods (where two of the double rods are not Premier League double rods) = ${}^{3}C_{1}$ (= 3)

Having dealt with the "Premier League double rod", now deal with the other two double rods.

The number of ways to arrange these two double rods in a row = ${}^{2}C_{1}$ (= 2)

Finally, note that there are two positions for the Premier League team for the double rod:



These two possibilities are:

That is, 2^1 ways.

So the number of outcomes each with one all-Premier League match = ${}^{3}C_{1} \times {}^{2}C_{1} \times 2^1$ (= 12)

Therefore, P (m = 1) = $\dfrac{{}^{3}C_{1} \times {}^{2}C_{1} \times 2^1}{{}^{6}C_{3}}$

Fill in the latest results:

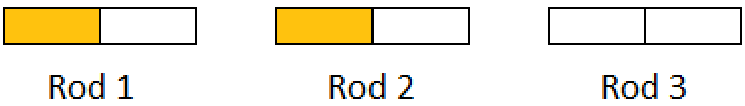
	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 0	1	0	0	0
t = 1	1	0	0	0
t = 2				
t = 3	$\frac{2^3}{^6C_3}$	$\frac{^3C_1 \times {}^2C_1 \times 2^1}{^6C_3}$	0	0
t = 4				
t = 5	0	0	1	0
t = 6	0	0	0	1

Find the probability distribution when **two** of the six teams are Premier League teams.

The total number of possible outcomes $= \dfrac{6!}{2!\,(6-2)!} = {}^6C_2$ (= 15)

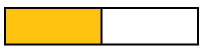
Find the number of ways to pick out the six teams so that the two Premier League teams avoid each other.

Set up three double rods as follows.



The number of ways to arrange these three double rods $= {}^3C_2$ (= 3)

Then note that there are two positions for each Premier League team for each of the double rods represented by:



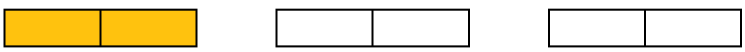
That is, 2^2 ways.

So the number of outcomes with no all-Premier League matches $= {}^3C_2 \times 2^2$ (= 12)

Therefore, $P(m = 0) = \dfrac{^3C_2 \times 2^2}{^6C_2}$

Find the number of ways to pick out the six teams so that the two Premier League teams have to play each other.

Set two of the Premier League teams together.



Treat these as three "double rods".

The number of ways to arrange the "Premier League double rod" in a row of three double rods $= {}^3C_1$ (= 3)

So the number of outcomes each with one all-Premier League match $= {}^3C_1$ (= 3)

Therefore, $P(m = 1) = \dfrac{^3C_1}{^6C_2}$

Fill in the latest results:

	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 0	1	0	0	0
t = 1	1	0	0	0
t = 2	$\dfrac{{}^3C_2 \times 2^2}{{}^6C_2}$	$\dfrac{{}^3C_1}{{}^6C_2}$	0	0
t = 3	$\dfrac{2^3}{{}^6C_3}$	$\dfrac{{}^3C_1 \times {}^2C_1 \times 2^1}{{}^6C_3}$	0	0
t = 4				
t = 5	0	0	1	0
t = 6	0	0	0	1

Find the probability distribution when **four** of the six teams are Premier League teams.

Firstly, note that if there are three matches and four Premier League teams then there must be at least one all-Premier League match. So it is impossible to have no all-Premier League matches in this case.

The total number of possible outcomes $= \dfrac{6!}{4!\,(6-4)!} = {}^6C_4$ (= 15)

[Note that ${}^6C_4 = {}^6C_2$]

Find the number of ways to pick out the six teams so that there is one all-Premier League match.

Set two of the Premier League teams together.

(Note: The other two premier League teams must be drawn separately so that there is only one all-Premier League match.)

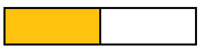
Treat these as three "double rods".

The number of ways to arrange the "Premier League double rod" in a row of three double rods (where two of the double rods are not Premier League double rods) $= {}^3C_1$ (= 3)

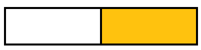
Having dealt with the "Premier League double rod", now deal with the other two double rods.

Note that there are two positions for the Premier League team for each of these double rods.

These two possibilities are:



or



This gives 2^2 extra ways.

So the number of outcomes each with one all-Premier League match $= {}^3C_1 \times 2^2$ (= 12)

Therefore, $P(m = 1) = \dfrac{{}^3C_1 \times 2^2}{{}^6C_3}$

Alternative Method

Find the number of ways to pick out the six teams so that there is one all-Premier League match.

Set two of the Premier League teams together.

(Note: The other two premier League teams must be drawn separately so that there is only one all-Premier League match.)

Symmetry can be used to help solve this problem.

Switch around the colours so that the orange rectangles become white and the white rectangles become orange.

So this problem is the same as finding the number of ways that two Premier League teams can be drawn apart when there are two Premier League teams and four non-Premier League teams in the draw.

This result has already been calculated as: ${}^{3}C_{2} \times 2^{2}$

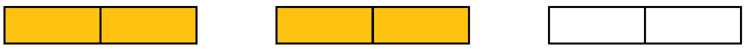
Therefore, $P(m = 1) = \dfrac{{}^{3}C_{2} \times 2^{2}}{{}^{6}C_{4}}$

[Note: Since ${}^{3}C_{2} \equiv {}^{3}C_{1}$ these methods produce the same result as required.

However, if ${}^{3}C_{1}$ is used then ${}^{3}C_{1}$ appears in all the nonzero probabilities in the second column of the probability distribution table. So using ${}^{3}C_{1}$ helps to identify patterns for the general case.]

Next, find the number of ways to pick out the six teams so that there are two all-Premier League matches.

Set two all-Premier League matches.



The number of ways to arrange these three “double rods” = ${}^{3}C_{2}$ (= 3)

Therefore, $P(m = 2) = \dfrac{{}^{3}C_{2}}{{}^{6}C_{4}}$

[Note: This probability must be the same as the probability of picking out one all-Premier League match when two of the six teams in the draw are Premier League teams.]

Finally, with only four Premier League teams in the draw it is not possible to have three all-Premier League matches which means that P (m = 3) = 0.

The completed probability distribution table for the probability of obtaining m all-premier league matches when there are t premier league teams in a draw for three matches is as follows:

	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 0	1	0	0	0
t = 1	1	0	0	0
t = 2	$\frac{{}^3C_2 \times 2^2}{{}^6C_2}$	$\frac{{}^3C_1}{{}^6C_2}$	0	0
t = 3	$\frac{2^3}{{}^6C_3}$	$\frac{{}^3C_1 \times {}^2C_1 \times 2^1}{{}^6C_3}$	0	0
t = 4	0	$\frac{{}^3C_1 \times 2^2}{{}^6C_4}$	$\frac{{}^3C_2}{{}^6C_4}$	0
t = 5	0	0	1	0
t = 6	0	0	0	1

Note that the sum of the probabilities in each row of the table must be 1.

The probabilities may now be given as fractions in their simplest form.

	P (m = 0)	P (m = 1)	P (m = 2)	P (m = 3)
t = 0	1	0	0	0
t = 1	1	0	0	0
t = 2	$\frac{4}{5}$	$\frac{1}{5}$	0	0
t = 3	$\frac{2}{5}$	$\frac{3}{5}$	0	0
t = 4	0	$\frac{4}{5}$	$\frac{1}{5}$	0
t = 5	0	0	1	0
t = 6	0	0	0	1

Now try the tutorial exercise for the quarter-final draw.

Use symmetry and look for patterns in the binomial coefficients to make the calculations more efficient.