Vulnavia

Ethereal Princess

The Road to Mora

Further Maths Tutorials

Vulnavia

Ethereal Princess

The Road to Mora is a programme of 24 tutorials which covers the AQA Further Maths Course.
The Spanish town of Mora is on the route from Madrid to the famous windmills at Consuegra.
So these tutorials prepare students for the A-Level programme entitled “The Windmills of Consuegra”.
This slideshow gives examples of topics from the first 15 tutorials.

Vulnavia presents

La Copa "El Dorado"

Ethereal Gold

You can enter for this maths cup by completing the exercises in this slideshow and the personal statement.

Obtain the official answer sheet from:

Chancellor@nightparklane.com

Further Maths Tutorial 1

Number

Example 1

Convert 0.123123123123123123 .... into a fraction.

Solution: Coming up!

Set $x = 0.123123123123123123\ ....$ [1]

Multiply by 10 to the power of n,

where n is the number of digits in the repeated block.

Set n = 3 since there are 3 digits (123) in the block.

So, $1000\,x = 123.123123123123\ ....$ [2]

Subtract [1] from [2] to get: $999\,x = 123$

So, $x = 123/999$

Tidy up to get: $x = \dfrac{41}{333}$

Number

Now it's your turn!

Exercise 1

Convert 0.123232323232323 ....

into a fraction in its simplest form.

Advice: Be careful when dealing with the 0.1 part.

Remember that you can always check your result using a calculator.

Write your result on the answer sheet.

Percentages, Ratio & Proportion

Example 2

In June 2016 the exchange rate for pounds to euros was

$$1 : x$$

By November, the exchange rate became 1 : 1.116

Given that the % reduction in the value of the pound compared to the euro was 20% find the value of x (as a decimal).

Solution: Coming up!

Solution

Be positive despite the percentage loss!

Think about what is left rather than what is lost.

A 20% reduction is equivalent to 80% remaining.

So 80% of x gives 1.116

Algebraically, $0.8x = 1.116$

Rearrange, $x = \frac{1.116}{0.8}$

This gives, $x = 1.395$

[So the original exchange rate was 1 : 1.395]

Exercise 2

The exchange rate for pounds to euros changes from 1 : 1.12 to 1 : 1.196.

Calculate the % increase in the value of the pound compared to the euro.

[Give the answer as a percentage to 2 decimal places.]

Write your result on the answer sheet.

Further Maths Tutorial 3

Manipulation of Surds

Example 3

Simplify the following expression.

$$\sqrt{\frac{1}{(\sqrt{2}-1)(\sqrt{2}+1)}}$$

Solution: Coming up!

$$(\sqrt{2}-1)(\sqrt{2}+1) = \sqrt{2}(\sqrt{2}+1) - (\sqrt{2}+1)$$

$$= 2 + \sqrt{2} - \sqrt{2} - 1 = 1$$

$$\sqrt{\frac{1}{(\sqrt{2}-1)(\sqrt{2}+1)}} = \sqrt{\frac{1}{1}} = 1$$

Manipulation of Surds

Exercise 3

Write the expression $\dfrac{4 + \sqrt{5}}{2 + \sqrt{5}}$ in the form $a + (b\sqrt{5})$

where a and b are integers.

Hint: Start by multiplying the numerator and denominator by $(2 - \sqrt{5})$.

Write your result on the answer sheet.

Further Maths Tutorial 4

Algebraic Manipulation

Example 4

Write the expression

$$\frac{3x - 1}{x^2 + 2x - 15} - \frac{1}{x - 3}$$

in the form: $\frac{a}{bx + c}$

where a, b and c are integers.

Solution: Coming up!

Algebraic Manipulation

Consider $$\frac{3x-1}{x^2+2x-15} - \frac{1}{x-3}$$

By factorisation: $x^2 + 2x - 15 \equiv (x+5)(x-3)$

$$\frac{3x-1}{(x+5)(x-3)} - \frac{1}{x-3} \equiv \frac{3x-1}{(x+5)(x-3)} - \frac{x+5}{(x+5)(x-3)}$$

$$\equiv \frac{3x-1 \quad - \quad (x+5)}{(x+5)(x-3)}$$

$$\equiv \frac{2x-6}{(x+5)(x-3)}$$

$$\equiv \frac{2(x-3)}{(x+5)(x-3)} \equiv \frac{2}{x+5}$$

[Note: a = 2, b = 1 and c = 5]

Algebraic Manipulation

Exercise 4

Write the expression

$$\frac{2x - 19}{2x^2 - 5x - 12} + \frac{1}{x - 4}$$

in the form: $$\frac{a}{bx + c}$$

where a, b and c are integers.

Write your result on the answer sheet.

Subject of a Formula

Example 5

Make n the subject of the formula.

$$p = \frac{3n + 2}{4n - 5}$$

Solution: Coming up!

Get n on its own!

Multiply through by $(4n - 5)$, $p(4n - 5) = 3n + 2$

So, $4pn - 5p = 3n + 2$

Get the n terms on one side, $4pn - 3n = 5p + 2$

Factorise the left-hand side, $(4p - 3)n = 5p + 2$

Divide through by $(4p - 3)$, $n = \frac{5p + 2}{4p - 3}$

Subject of a Formula

Exercise 5

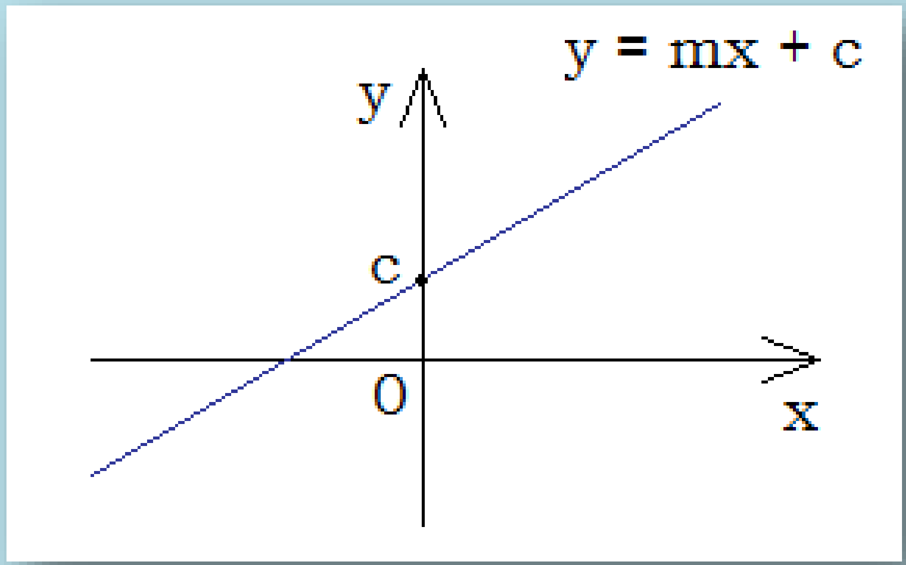
Make n the subject of the formula.

$$c = \sqrt{\frac{4n + 3}{2 + 5n}}$$

Write your result on the answer sheet.

Further Maths Tutorial 6

The Straight Line



The equation of a straight line of gradient m

and y-intercept c is: $y = mx + c$

Parallel lines have the same gradient.

The product of the gradients of perpendicular lines is -1

The Straight Line

Example 6

Find the equation of the straight line that is perpendicular to the line with the equation: $y = 2x + 5$ and passes through the point (2, 1).

Write the answer in the form: $ax + by + c = 0$ where a, b and c are integers.

Solution: Coming up!

The Straight Line

Solution

The product of the gradients of perpendicular lines is -1.

So, $m_1 m_2 = -1$ Taking $m_1 = 2$ (the gradient of the given line) gives $m_2 = -1/2$

The equation of the required line has the form:

$$y = -\frac{1}{2}x + c$$

The line passes through (2, 1) so: $1 = -1 + c$ Thus, $c = 2$

So the equation of the line is: $y = -\frac{1}{2}x + 2$

Multiply through by 2 and rearrange to get: $x + 2y - 4 = 0$

[Note that $a = 1$, $b = 2$ and $c = -4$.]

The Straight Line

Exercise 6

The straight line L is parallel to the line $y = 4x - 2$ and passes through the point (2, 5).
Find the equation of the straight line which passes through the point (8, 2) and is perpendicular to line L.

Write the answer in the form: $ax + by + c = 0$
where a, b and c are integers.

Write your result on the answer sheet.

Further Maths Tutorial 7

Sequences Portfolio

Example 7

Use the method of differences to find the generating formula of the triangle number sequence:

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad \ldots.$$

Solution: Coming up!

Write out the first row of differences.

1	3	6	10	15	21	....
2	3	4	5	6		

Next, write out the second row of differences.

1	3	6	10	15	21	....
2	3	4	5	6		
1	1	1	1			

Sequences Portfolio

Solution: Continued …

1 3 6 10 15 21 ….

2 3 4 5 6

1 1 1 1

As there is a constant gap in the second row of differences the given sequence must be a quadratic sequence.

The quadratic term in the formula is given by:

$$\left(\frac{\text{constant gap}}{2!}\right) n^2 = \frac{1}{2} n^2$$

Sequences Portfolio

Solution: Continued …

Next, compare the triangle number sequence with the terms in the sequence generated by $n^2/2$.

S	1	3	6	10	15	21	….
$\frac{1}{2}n^2$	$\frac{1}{2}$	2	$\frac{9}{2}$	8	$\frac{25}{2}$	18	
Linear part	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	

Add a linear part to each quadratic term to obtain the corresponding terms in the triangle number sequence.

The formula for the linear part is: $n/2$

So the triangle number formula is: $S = \frac{n^2}{2} + \frac{n}{2}$

Sequences Portfolio

Exercise 7

According to the song "The Twelve Days of Christmas", what is the total number of presents that you receive from your true love over Christmas?

[Note: Count a partridge in a pear tree as 1 present!]

Write your result on the answer sheet.

Sequences Portfolio

Investigation: The n Days of Christmas

Suppose that the song carries on for n days of Christmas.

Devise a formula which gives the total number of presents P received after n days.

Solving Quadratic Equations

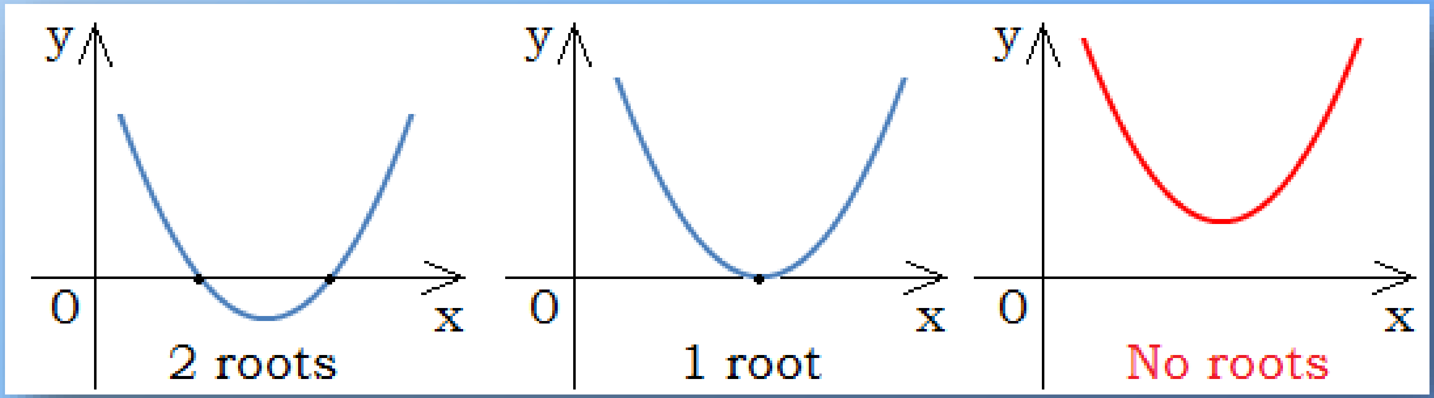
The general form of a quadratic equation is:

$$ax^2 + bx + c = 0$$

The equation of the parabola is: $y = ax^2 + bx + c$

So the roots of the equation: $ax^2 + bx + c = 0$

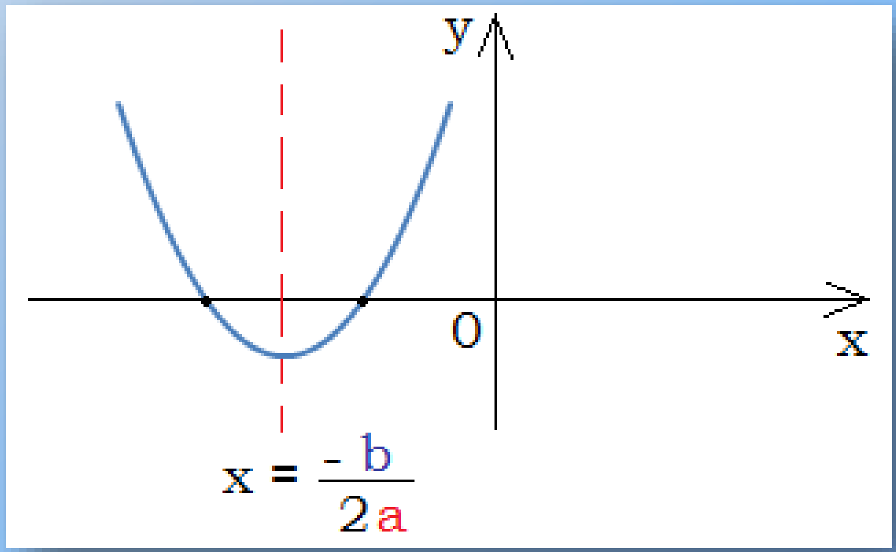
represent the x-coordinates of the points where the parabola crosses the x-axis.



Solving Quadratic Equations

An important property of the parabola can be used to produce the general formula to solve a quadratic equation.

All parabolas have a line of symmetry at: $x = \frac{-b}{2a}$

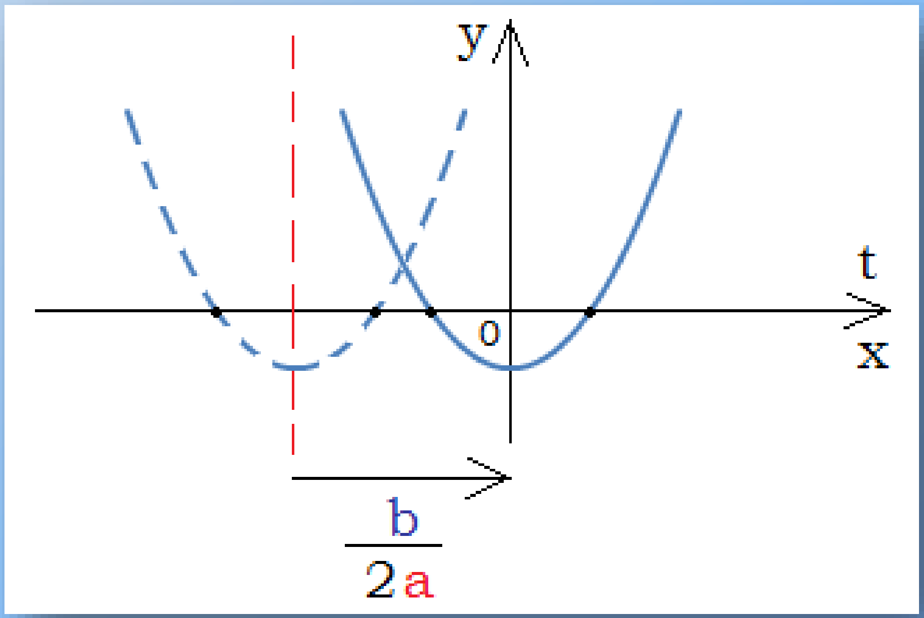


Shift the parabola to the right by $\frac{b}{2a}$ to make the y-axis the line of symmetry.

Solving Quadratic Equations

To shift a curve to the right, by a distance d, apply the transform $x \longrightarrow (t - d)$

In this case set: $d = \dfrac{b}{2a}$



The parabola is shifted to a coordinate system where $t = 0$ is the line of symmetry.

Solving Quadratic Equations

To see how the shifted parabola can be used to solve the general quadratic equation, set $x \rightarrow (t - \frac{b}{2a})$

in the quadratic equation: $ax^2 + bx + c = 0$

$$a\left(t - \frac{b}{2a}\right)^2 + b\left(t - \frac{b}{2a}\right) + c = 0$$

$$a\left(t - \frac{b}{2a}\right)\left(t - \frac{b}{2a}\right) + b\left(t - \frac{b}{2a}\right) + c = 0$$

$$a\left(t^2 - \frac{b}{a}t + \frac{b^2}{4a^2}\right) + bt - \frac{b^2}{2a} + c = 0$$

$$at^2 - \cancel{bt} + \frac{b^2}{4a} + \cancel{bt} - \frac{b^2}{2a} + c = 0$$

Divide through by a, $t^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

Solving Quadratic Equations

Recap: $t^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

Write the equation as: $t^2 = \frac{b^2}{4a^2} - \frac{ac}{a^2}$

So, $t^2 = \frac{b^2 - 4ac}{4a^2}$

Square root both sides: $t = \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$

Finally, find x by using the transform: $x = t - \frac{b}{2a}$

Thus, $x = -\frac{b}{2a} \pm \frac{\sqrt{(b^2 - 4ac)}}{2a}$

So, $x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$

This is the quadratic equation formula.

Further Maths Tutorial 8

Solving Quadratic Equations

The quadratic equation formula is:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

The discriminant is: $b^2 - 4ac$

If $b^2 - 4ac > 0$ the quadratic equation has 2 real roots.

If $b^2 - 4ac = 0$ the quadratic equation has equal roots.

If $b^2 - 4ac < 0$ the quadratic equation has no real roots.

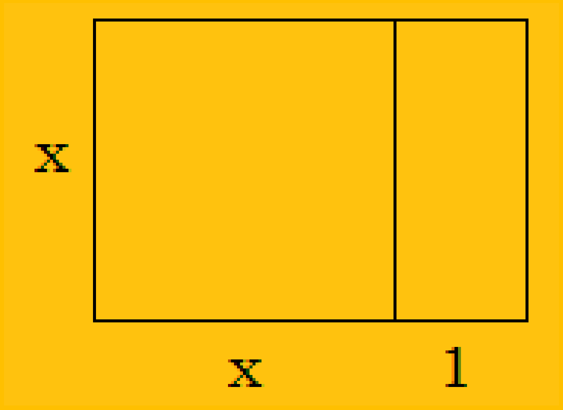
Note that if a and c have opposite signs then the equation must have real roots.

If $b^2 - 4ac$ is a square number then the quadratic equation may be factorised.

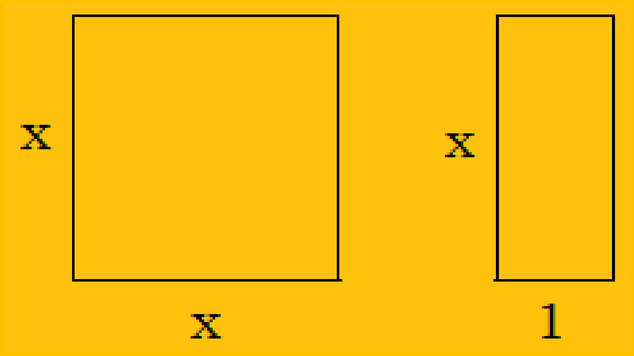
Further Maths Tutorial 8

Solving Quadratic Equations

Consider the following rectangle (x > 1 case shown).



Cut-out a square of side x from the rectangle.



For the golden rectangle: $\frac{x+1}{x} = \frac{x}{1}$

Solving Quadratic Equations

Recap: $\dfrac{x+1}{x} = \dfrac{x}{1}$

Cross-multiply: $x + 1 = x^2$

Rearrange to get: $x^2 - x - 1 = 0$

Compare this quadratic equation with the general quadratic equation: $a x^2 + b x + c = 0$

So, $a = 1$, $b = -1$ and $c = -1$

Use the quadratic equation formula:

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(\{-1\}^2 - 4(1)\{-1\})}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

For $x > 0$, the golden ratio is: $\left(\dfrac{1 + \sqrt{5}}{2}\right) : 1$

Solving Quadratic Equations

Exercise 8

The golden ratio may also be written in the form 1 : x where $x < 0$.

Establish the quadratic equation which has to be solved in this case to give the golden ratio.

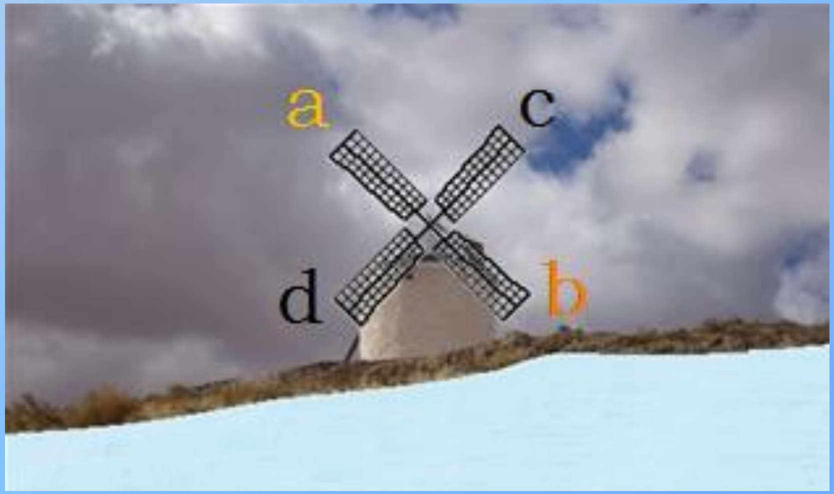
Write your equation on the answer sheet.

Solving Simultaneous Equations

This tutorial looks at a magical method to solve simultaneous equations without using algebraic manipulation.

Consider the concept of the determinant.

$$\begin{vmatrix} a & c \\ d & b \end{vmatrix} = ab - cd$$



Solving Simultaneous Equations

Consider the simultaneous linear equations:

$$ax + cy = u$$
$$dx + by = v$$

It can be shown that:

$$x = \frac{-\begin{vmatrix} c & u \\ b & v \end{vmatrix}}{\begin{vmatrix} a & c \\ d & b \end{vmatrix}}$$ (Block out x terms.)

$$y = \frac{\begin{vmatrix} a & u \\ d & v \end{vmatrix}}{\begin{vmatrix} a & c \\ d & b \end{vmatrix}}$$ (Block out y terms.)

Notice that the denominators of x and y are equal.

Further Maths Tutorial 9

Solving Simultaneous Equations

Example 9

Solve the simultaneous linear equations:

$$2x + 3y = 3$$
$$6x + 5y = 1$$

Using determinants:

$$x = \frac{-\begin{vmatrix} 3 & 3 \\ 5 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 6 & 5 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 2 & 3 \\ 6 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 6 & 5 \end{vmatrix}}$$

$$= \frac{-(3 - 15)}{10 - 18} = \frac{12}{-8} \qquad = \frac{(2 - 18)}{10 - 18} = \frac{-16}{-8}$$

Therefore, $x = -1.5$ and $y = 2$

Solving Simultaneous Equations

Exercise 9: Use the determinant method to solve the simultaneous linear equations:

$$4x + 8y = 9$$
$$3x + 5y = 3$$

(i) Work out $-\begin{vmatrix} 8 & 9 \\ 5 & 3 \end{vmatrix}$

(ii) Work out $\begin{vmatrix} 4 & 9 \\ 3 & 3 \end{vmatrix}$

(iii) Work out $\begin{vmatrix} 4 & 8 \\ 3 & 5 \end{vmatrix}$

(iv) Use these results to find x and y.

Write your results on the answer sheet.

Domain and Range

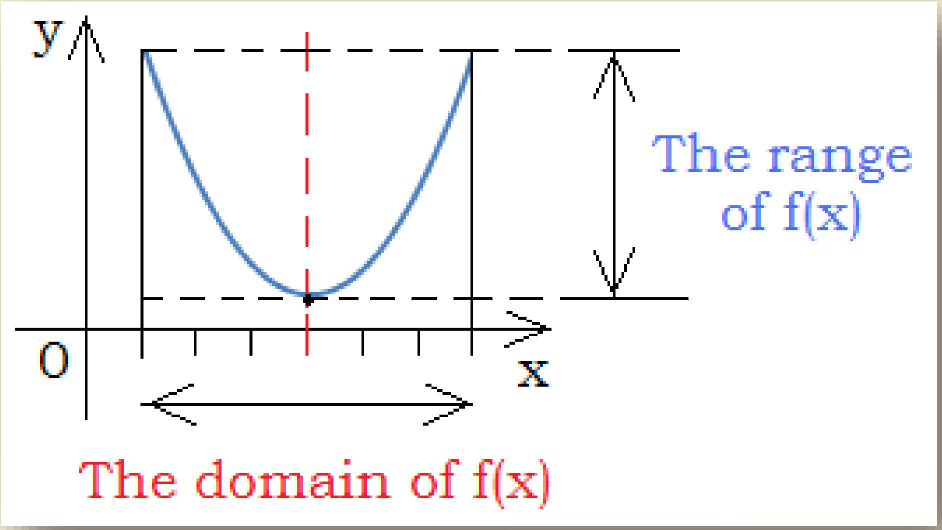
Consider the function y = f(x).

The domain of this function is the interval of x values which are under consideration.

For example, $x > 0$ or $-1 \leq x < 1$

The range of this function is the interval of y values which corresponds to the x-values of the domain.

Consider a parabola y = f(x).



Further Maths Tutorial 10

Domain and Range

Example 10

Find the range of the quadratic function: $f(x) = 1 + 4x - x^2$ for the domain $x > 1$.

Solution: Coming up!

Consider the parabola $y = 1 + 4x - x^2$

The sign of the number in front of x^2 determines the type of parabola.

If this number (a) is positive then the parabola is ∪-type so that the parabola has a minimum point.

If the number (a) is negative then the parabola is ∩-type so in this case the parabola has a maximum point.

The turning point of a parabola (minimum or maximum) is always on the line of symmetry and the equation of the line of symmetry of the parabola $y = ax^2 + bx + c$ is:

$$x = \frac{-b}{2a}$$

Domain and Range

Solution: Continued ...

Find the equation of the line of symmetry of the given parabola: $y = 1 + 4x - x^2$

Comparing with the general parabola: $y = ax^2 + bx + c$ gives: $a = -1$, $b = 4$, $c = 1$.

Since $a < 0$ this parabola is ∩-type so it has a maximum turning point.

The equation of the line of symmetry is:

$$x = \frac{-b}{2a} = \frac{-4}{2\{-1\}} = 2 \qquad \text{That is, } x = 2$$

The value $x = 2$ is within the domain $x > 1$.

When $x = 2$, $y = 1 + 4(2) - (2)^2 = 5$

So the maximum point of this parabola is at: $\{2, 5\}$

Domain and Range

Recap: The maximum point of the parabola:

$$y = 1 + 4x - x^2 \text{ is at } \{2, 5\}$$

Since the x-coordinate of the maximum point is within the domain $x > 1$ the upper limit of the range is the y-coordinate of the maximum point. That is, $y = 5$.

For the domain $x > 1$ there is no lower limit for y. (The parabola curve will keep going downwards as x increases beyond $x = 2$.)

So the range of the quadratic function for the domain $x > 1$ is:

$$f(x) \leq 5$$

Domain and Range

Exercise 10: Find the range of the quadratic function:
$f(x) = x^2 - 8x + 2$
for the domain $0 \le x \le 6$.

Note: As the domain has upper and lower limits the range of f(x) will have upper and lower limits.

Write your result on the answer sheet.

Graph Sketching / Plotting

The graph drawing tutorial examines graph sketching and graph plotting.

When a sketch is required the essential features of the graph must be illustrated. These features could be coordinates of the intersection points of the graph with the axes, coordinates of turning points, behaviour as $x \to \pm \infty$ or aspects of symmetry.

To plot a graph a particular step value is used for x then corresponding y-values are calculated.

The points are then joined and any key features included as well.

Further Maths Tutorial 11

Graph Sketching / Plotting

Example 11

Sketch the graphs of $y = x^2 - 6x + 8$ and $y = x - 1$.
Calculate the coordinates of the intersection points of the parabola and the straight line.

Solution: Coming up!

The equation of the line of symmetry of the parabola is:

$x = -\frac{\{-6\}}{2} = 3$. That is, $x = 3$.

When $x = 3$, $y = 3^2 - 6(3) + 8 = -1$

So the parabola has a minimum at $\{3, -1\}$

Now, since the equation of the x-axis is $y = 0$,
any intersection points of the parabola with the x-axis can be found by solving the equation $x^2 - 6x + 8 = 0$.

Factorise to get: $(x - 2)(x - 4) = 0$ So, $x = 2$ or $x = 4$.

The intersection points with the x-axis are: (2, 0) & (4, 0)

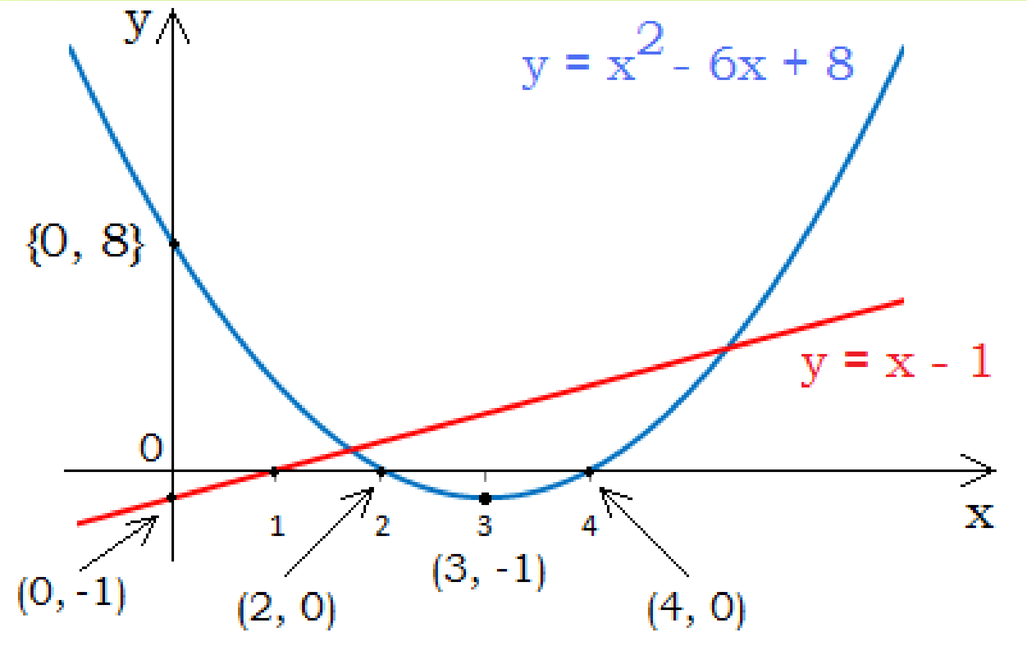
Graph Sketching / Plotting

Solution: Continued ...

The equation $y = x - 1$ represents a straight line of gradient 1 and y-intercept -1.
So the line crosses the y-axis at (0, -1}

Note that this line crosses the x-axis at (1, 0).

The following diagram shows the parabola and the line.



Notice the different scale on the x and y-axes.

Graph Sketching / Plotting

Solution: Continued ...

Now find the coordinates of the intersection points of the parabola $y = x^2 - 6x + 8$ and the straight line $y = x - 1$.

Equate the equation of the parabola and the equation of the line to find the x-coordinates of the intersection points.

At the intersection points, $x^2 - 6x + 8 = x - 1$

Rearrange into the general form of a quadratic equation:

Subtract x from both sides: $x^2 - 7x + 8 = -1$

Add 1 to both sides: $x^2 - 7x + 9 = 0$

Graph Sketching / Plotting

Exercise 11:

(a) Use the quadratic equation formula to solve the equation: $x^2 - 7x + 9 = 0$

(b) Use these x values to find the corresponding y-coordinates of the intersection points of the parabola $y = x^2 - 6x + 8$ and the straight line $y = x - 1$.

Write the coordinates of the intersection points on the answer sheet to 3 significant figures.

Further Maths Tutorial 12

Matrices Portfolio

This tutorial involves the study of matrices to perform transformations in 2D.

Transformation matrices may be derived by using the unit matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

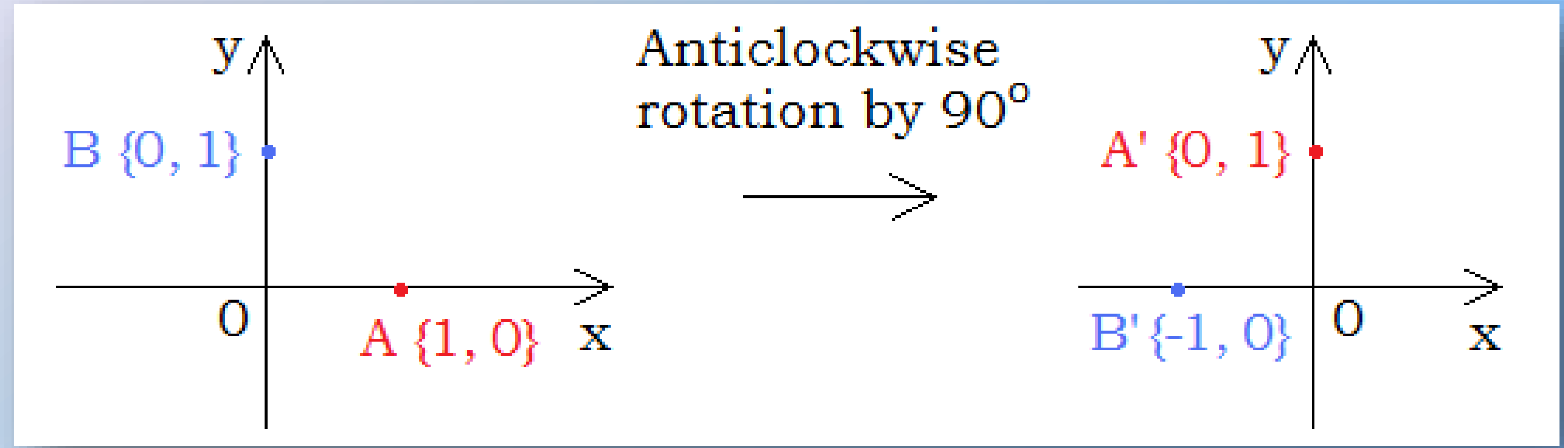
Consider the position vectors of two points A and B contained in the unit matrix.

$$\begin{matrix} \text{A} & \text{B} \\ \end{matrix}$$
$$\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

Transform the points A {1, 0} and B {0, 1} to find the coordinates of the transformed points A' and B'.

Matrices Portfolio

Rotate the points A and B by 90° anticlockwise about the origin.



Thus, A → A' {0, 1} and B → B' {-1, 0}.

So the transformation matrix for anticlockwise rotation by 90 degrees about the origin is:

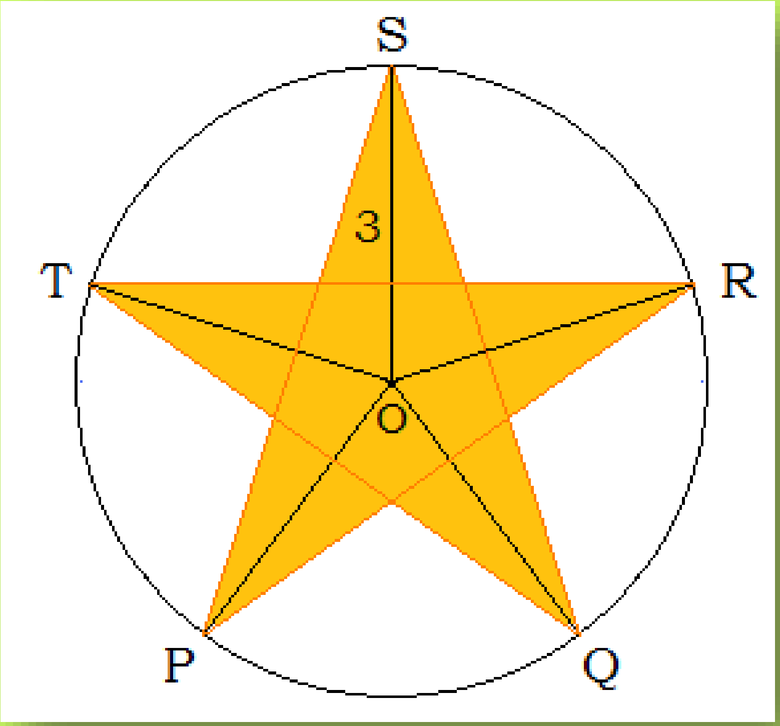
$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This matrix can then be applied to the position vector of any point to find the transformed point.

Matrices Portfolio

Example 12

Now consider an application to art.



Rotate the star anticlockwise by 90º about the origin O.

Note the coordinates:

P {-3 cos 54º, -3 sin 54º }, Q {3 cos 54º, -3 sin 54º },
R { 3 cos 18º, 3 sin 18º }, S {0, 3} and T {-3 cos 18º, 3 sin 18º}

Further Maths Tutorial 12

Matrices Portfolio

Rotate the point P {-3 cos 54°, -3 sin 54°} anticlockwise by 90 degrees about the origin.

Use the transformation matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overset{P}{\begin{pmatrix} -3\cos 54^\circ \\ -3\sin 54^\circ \end{pmatrix}} = \begin{pmatrix} 0 + 3\sin 54^\circ \\ -3\cos 54^\circ + 0 \end{pmatrix} = \overset{P'}{\begin{pmatrix} 3\sin 54^\circ \\ -3\cos 54^\circ \end{pmatrix}}$$

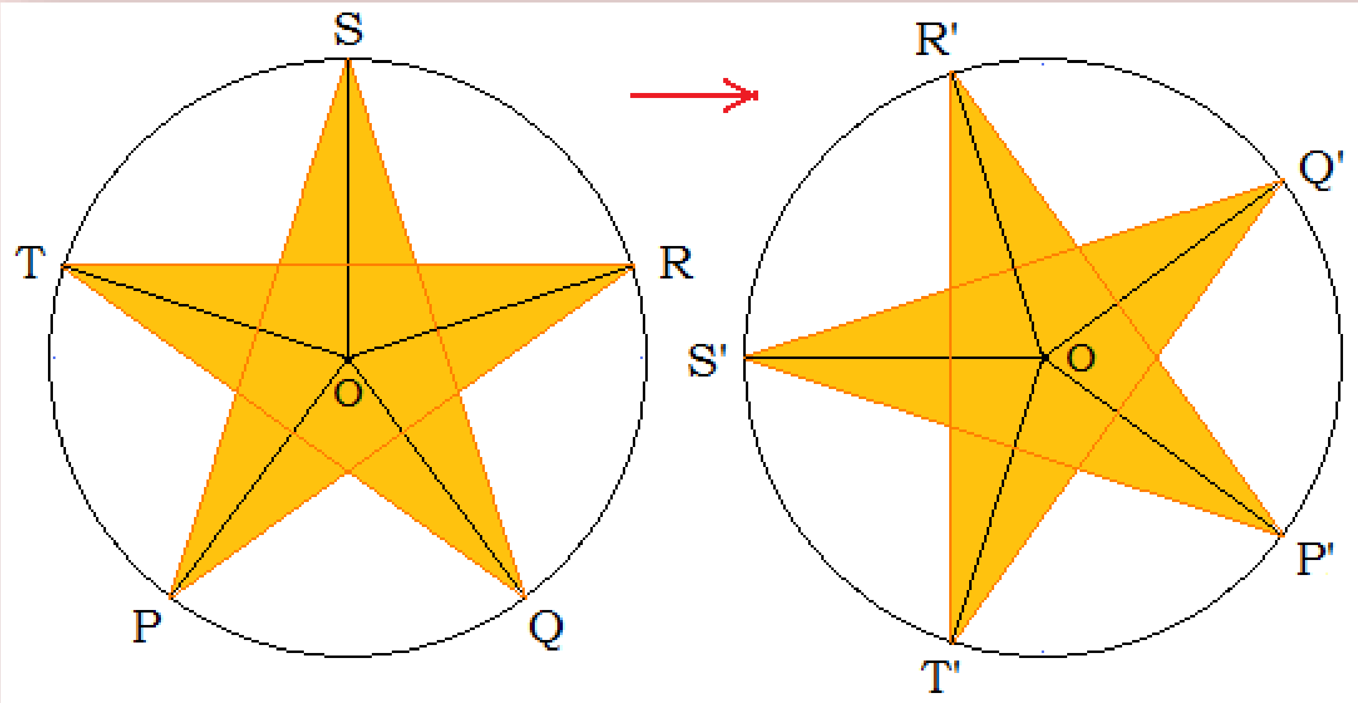
So the coordinates of the point P' are:

{ 3 sin 54°, -3 cos 54°}

Further Maths Tutorial 12

Matrices Portfolio

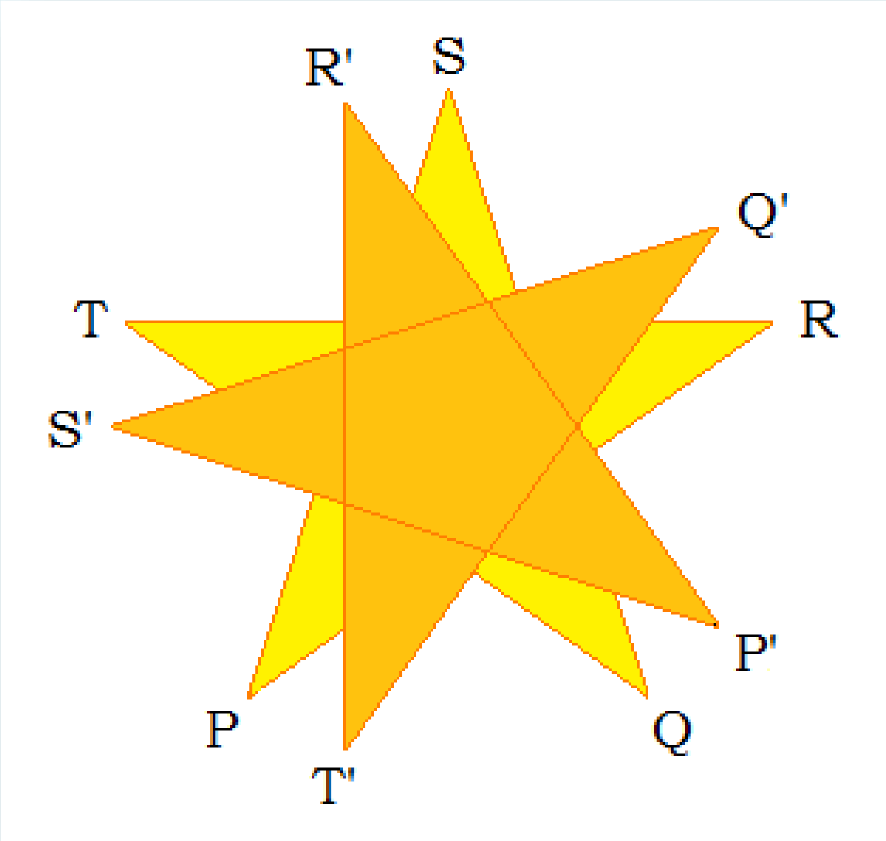
Exercise 12:



Use the transformation matrix to find the coordinates of the other transformed points Q’, R’, S’ and T’.

Write the coordinates, in trigonometric form, on the answer sheet.

Matrices Portfolio



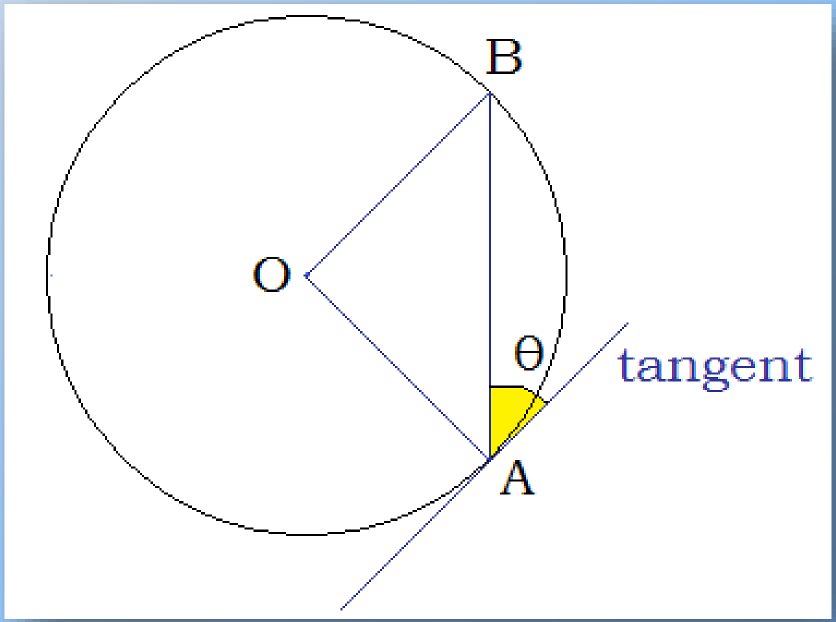
Matrices can be applied to produce precise artistic drawings.

Further Maths Tutorial 13

Circle Theorems

Derivation of the "Alternate Segment Theorem".

Start by considering the triangle OAB in a circle, centre O.



Introduce the tangent to the circle at the point A.

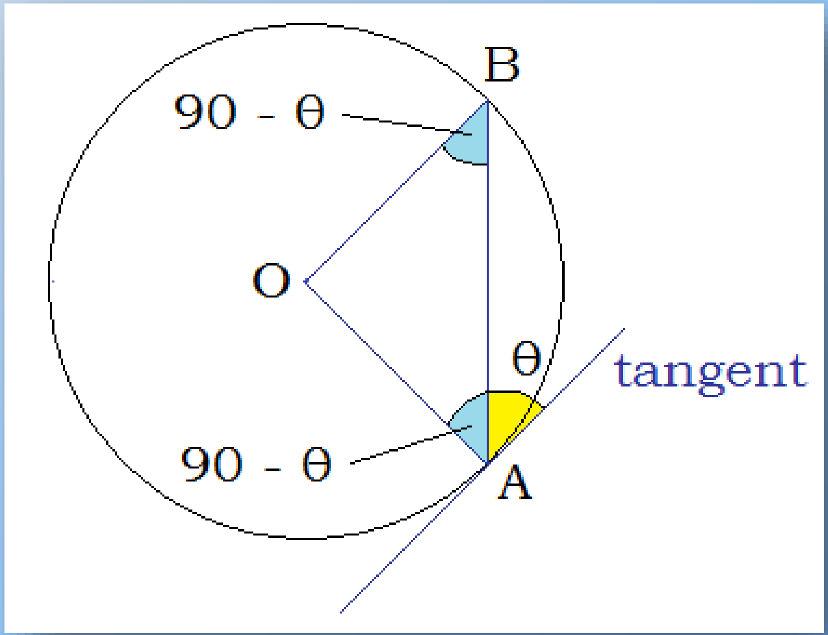
Label the angle between this tangent and the chord AB as θ.

Further Maths Tutorial 13

Circle Theorems

Derivation of the "Alternate Segment Theorem".

Since the angle between a tangent and a radius at a point on the circle is 90 degrees, the angle OAB = 90 - θ.



Since OA = OB the triangle AOB is isosceles.

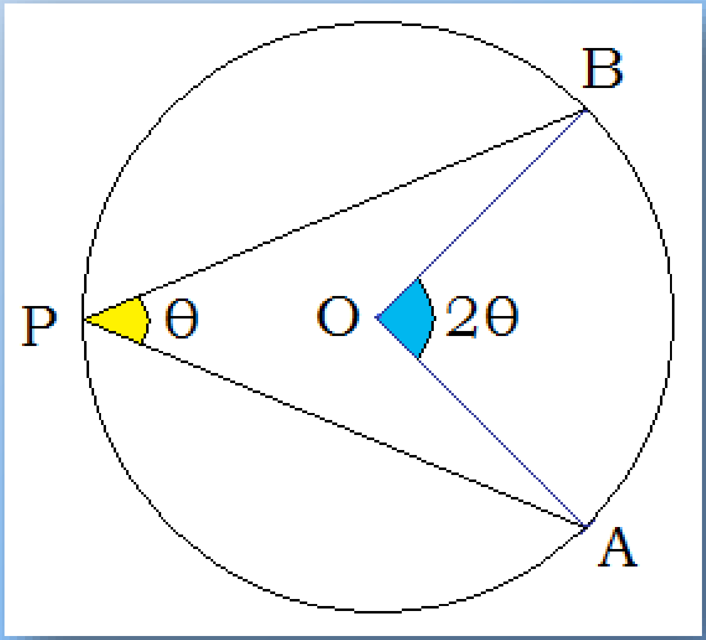
So the angle OBA is also 90 - θ.

Now, angle AOB = 180 – (90 - θ) - (90 - θ) = 2θ

Circle Theorems

Derivation of the "Alternate Segment Theorem".

By the circle theorem "Angle at the Circumference", the angle APB at the circumference is half the angle AOB at the centre.



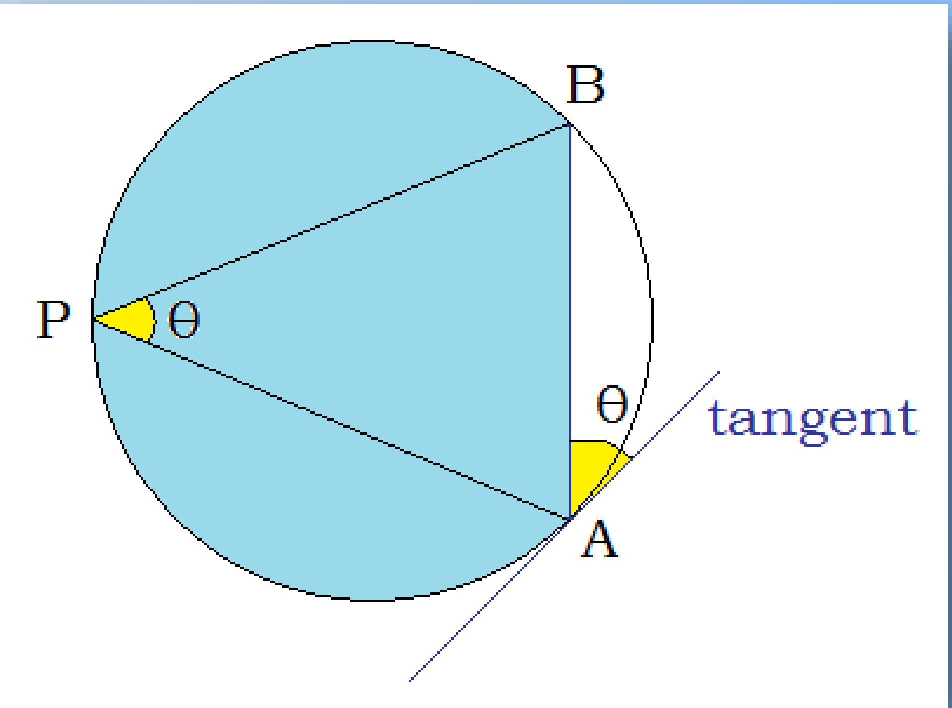
So the angle APB = θ

[Note: P is any point on the major arc AB.]

Circle Theorems

Derivation of the "Alternate Segment Theorem".

So the angle between the chord AB and the tangent at point A is equal to the angle at the circumference at point P (where P is any point on the major arc AB).



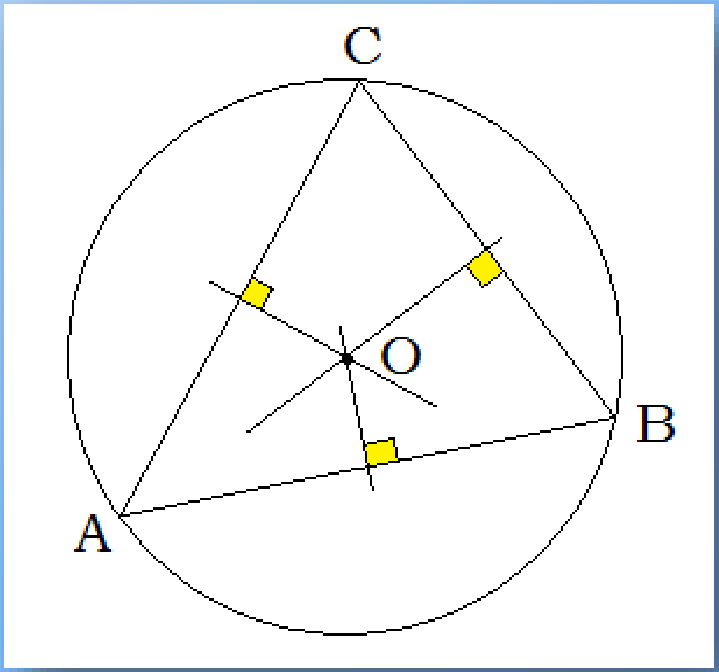
This is the Alternate Segment Theorem.

Circle Theorems

Triangles in a Circle.

It is always possible to draw a circle through the vertices of a triangle.

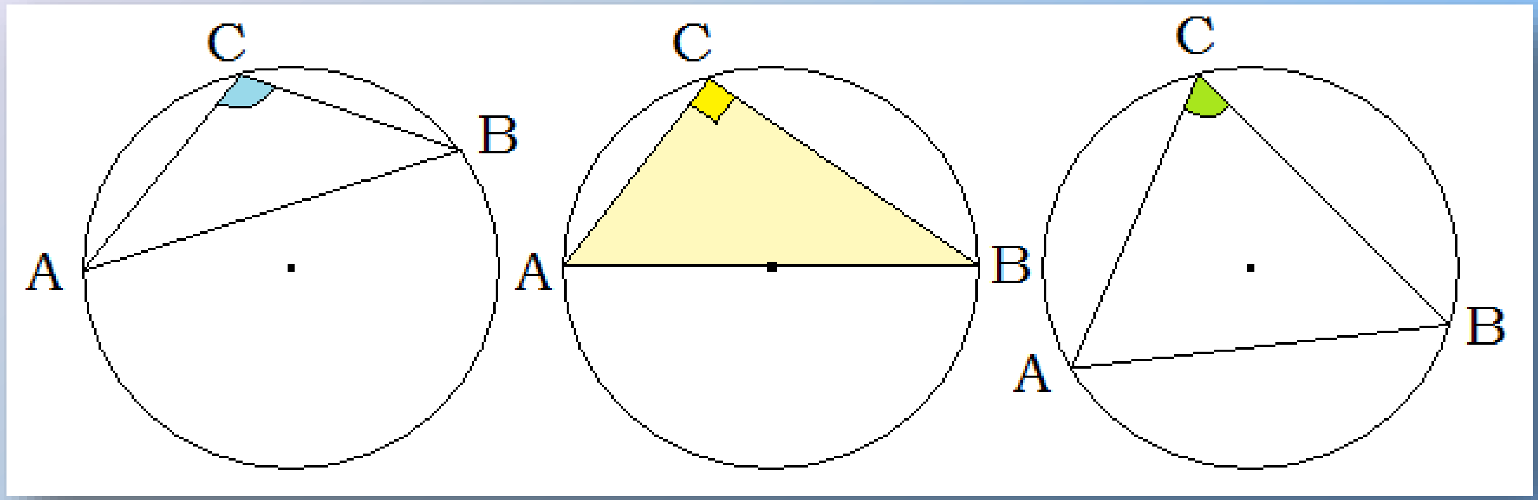
Regard the sides of the triangle as chords of a circle. Since a radius bisects a chord at 90^o the perpendicular bisectors of the sides of the triangle must meet at the centre of the circle which passes through the vertices of the triangle.



Further Maths Tutorial 13

Circle Theorems

Triangles in a Circle.



If the angle ACB is obtuse then the centre of the circle through the vertices of the triangle is outside the triangle ABC.

If the angle ACB is a right-angle then the situation becomes "The Angle in a Semicircle" so that the centre of the circle through the triangle ABC is at the midpoint of the diameter AB.

If all three angles of triangle ABC are acute then the centre of the circle through the vertices of the triangle is inside the triangle ABC.

Further Maths Tutorial 13

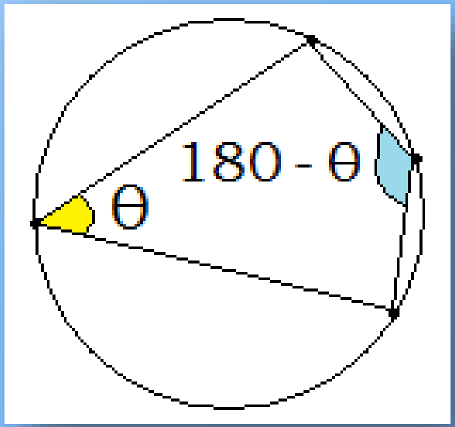
Circle Theorems

Cyclic Quadrilateral.

Unlike the case with triangles, it is not always possible to draw a circle through the vertices of a quadrilateral.

How can you tell whether or not it is possible to draw a circle through the vertices of a quadrilateral just by using the interior angles of the quadrilateral?

It turns out that if the opposite angles of the quadrilateral add up to 180° then a circle can be drawn through its vertices. Otherwise a circle cannot be drawn through the vertices of the quadrilateral.



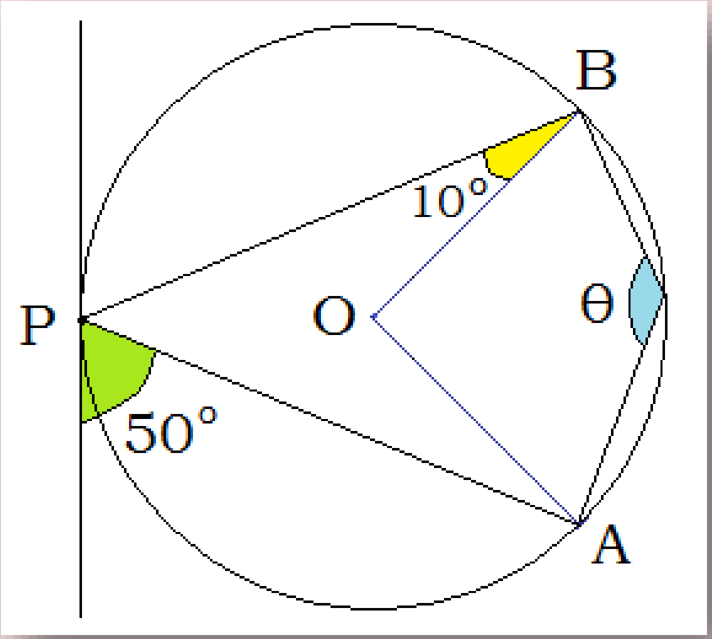
This is a cyclic quadrilateral.

Further Maths Tutorial 13

Circle Theorems

Exercise 13

The angle between the tangent to the circle at P and the chord PA is 50° and O is the centre of the circle through A, B and P.



Calculate the angle θ.

Hint: Add guidelines to the diagram to help.
"When in doubt think about isosceles triangles in a circle."

Write your result on the answer sheet.

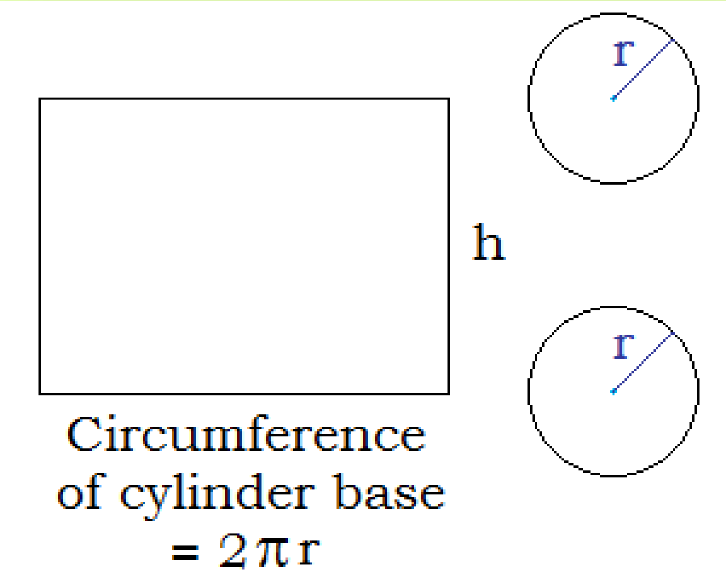
Mensuration

Example 14

This tutorial covers calculations which involve length, area and volume.

Find the height h of a cylinder of radius r so that the total surface area of the cylinder (with two circular ends) is the same as the total surface area of a hemisphere with a flat circular base of radius r.

Solution: Coming up!



The curved surface area of the cylinder = $2\pi rh$.

The area of the two ends = $2(\pi r^2)$

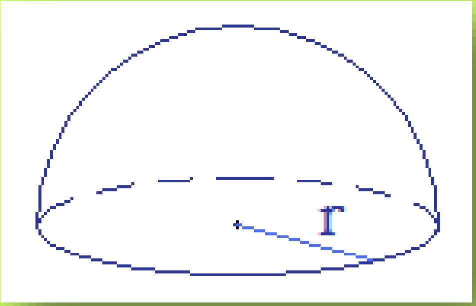
So the total surface area of the cylinder

$= 2\pi rh + 2\pi r^2$

$= 2\pi r(h + r)$

Mensuration

The surface area of a sphere of radius r is $4\pi r^2$.



So the curved surface area of the hemisphere of radius r is $2\pi r^2$.

The area of the flat base = πr^2.

So the total surface area of the hemispherical shape
$= 2\pi r^2 + \pi r^2$
$= 3\pi r^2$

Mensuration

Recap: The total surface area of the cylinder = $2\pi r\,(h + r)$

The total surface area of the hemisphere = $3\pi r^2$

If these areas are equal then:

$$2\pi r\,(h + r) = 3\pi r^2$$

$$2\pi r h + 2\pi r^2 = 3\pi r^2$$

$$2\pi r h = \pi r^2$$

Divide through by $2\pi r$.

So, $h = \frac{r}{2}$

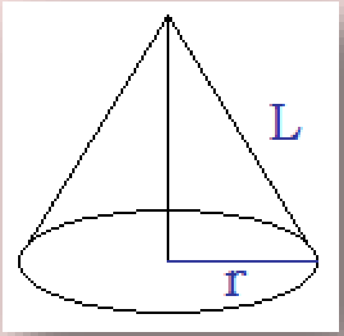
Therefore, for the total surface area of the cylinder to be equal to the total surface area of the hemispherical shape:

$$h = \frac{r}{2}$$

Mensuration

Exercise 14

The total surface area of a cone with a flat circular base of radius r is given as π (square units).



(a) Given that the curved surface area of a cone of slant height L and base radius r is: $\pi r L$ and that in this case L = 1 find the radius of the base of the cone.
[Leave your answer in surd form.]

(b) Comment on the ratio L : r in this case.

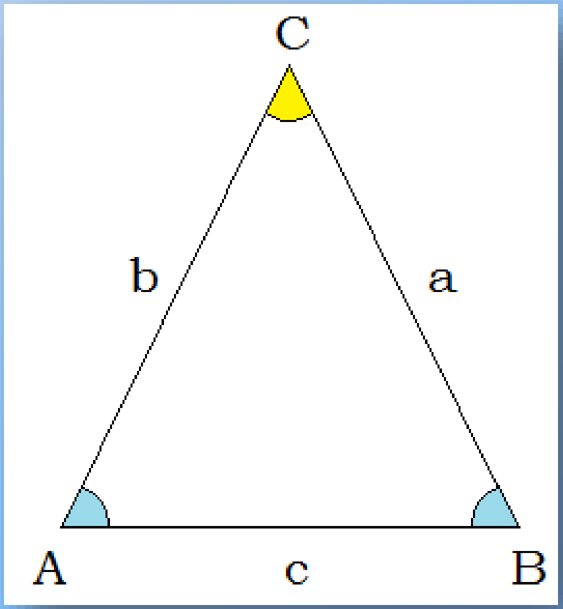
Write your results on the answer sheet.

The Sine and Cosine Rules

Derivation of the Sine Rule

The sine rule, applied to the triangle ABC, is:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

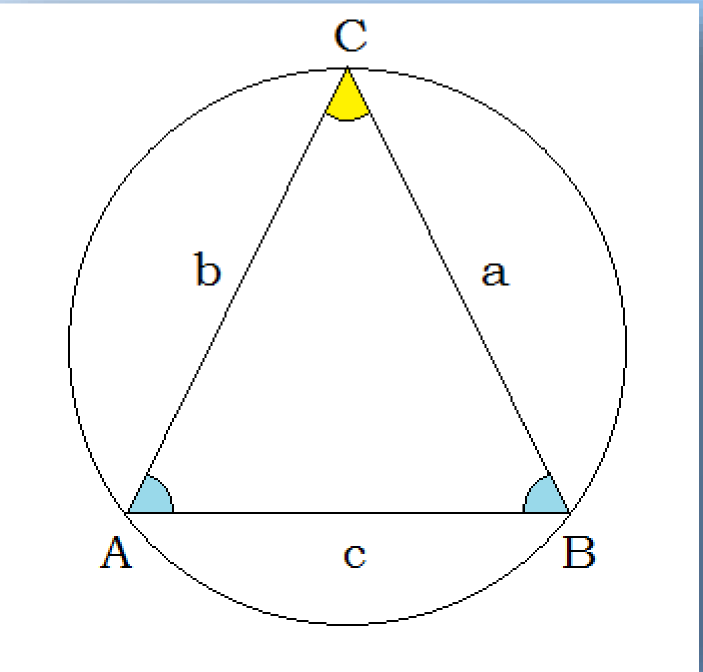


The purpose of this exploration is to find out what geometrical feature the sine rule ratio represents.

The Sine and Cosine Rules

Derivation of the Sine Rule

This derivation involves "thinking outside the triangle"!

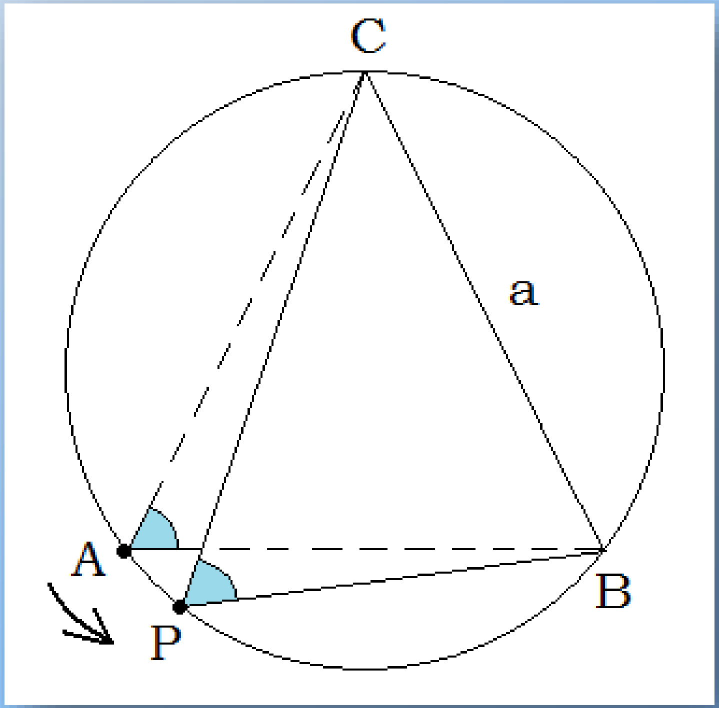


Consider the circle drawn through the vertices of the triangle ABC.

The Sine and Cosine Rules

Derivation of the Sine Rule

Now apply the circle theorem "Angles in the Same Segment".

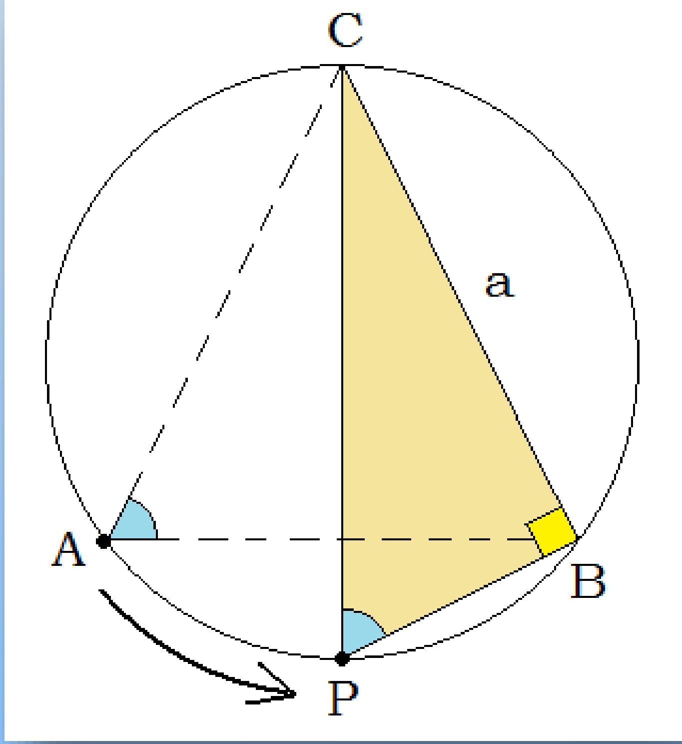


The point P (initially at point A) can move anywhere around the major arc BC without changing the angle BPC. Clearly, the chord length BC remains the same as well.

The Sine and Cosine Rules

Derivation of the Sine Rule

Since the point P can move anywhere on the major arc BC fix it in the position where angle CBP is a right-angle.

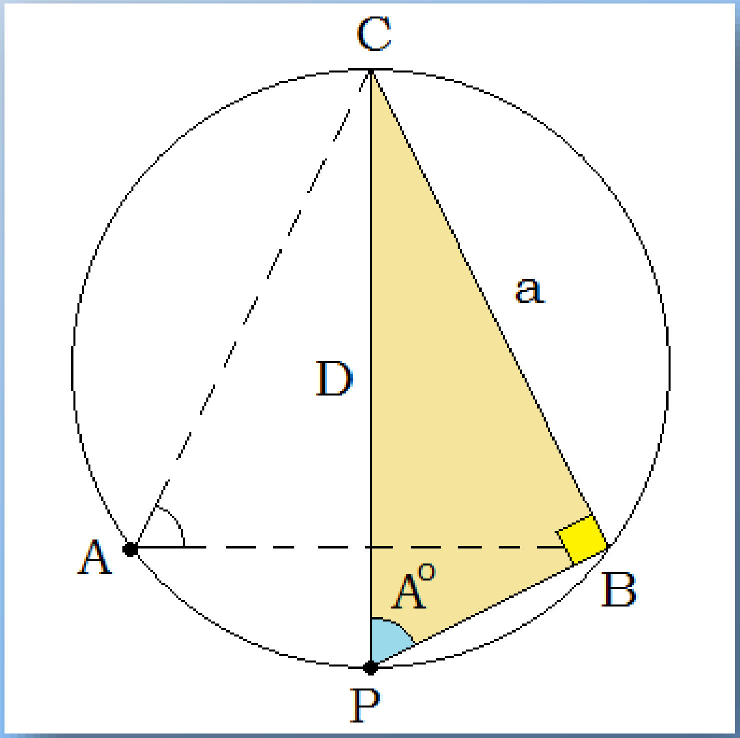


In this situation another circle theorem comes into play: "Angle in a Semicircle". Here, CP is the diameter of the circle.

The Sine and Cosine Rules

Derivation of the Sine Rule

Apply the sine ratio to the right-angled triangle BPC.
The diameter D of the circle is the hypotenuse of triangle BPC.

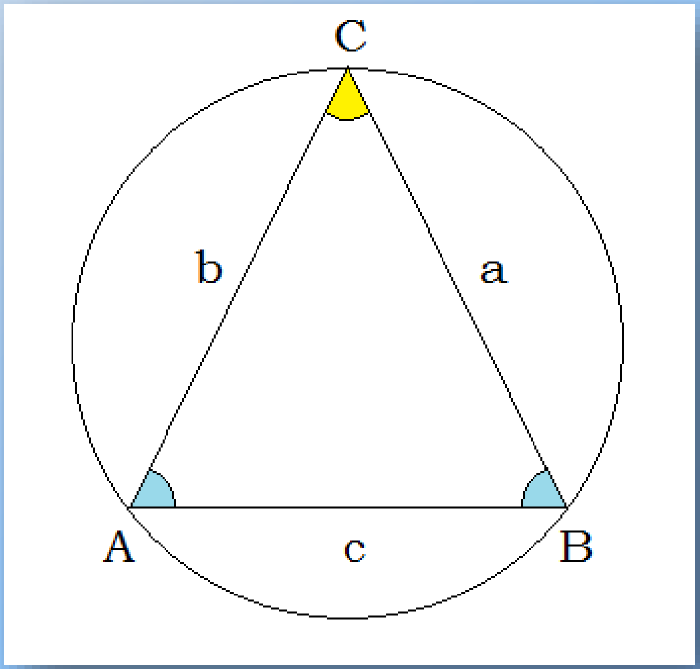


$$\frac{a}{\sin A} = D$$

The Sine and Cosine Rules

Derivation of the Sine Rule

Similarly, it can be shown that $\dfrac{b}{\sin B} = D$ and $\dfrac{c}{\sin C} = D$



Therefore, $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = D$$

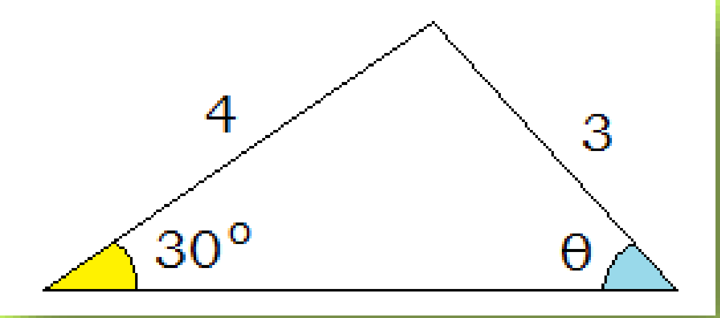
where D is the diameter of the circle through points A, B & C.

Further Maths Tutorial 15

Example 15a(i)

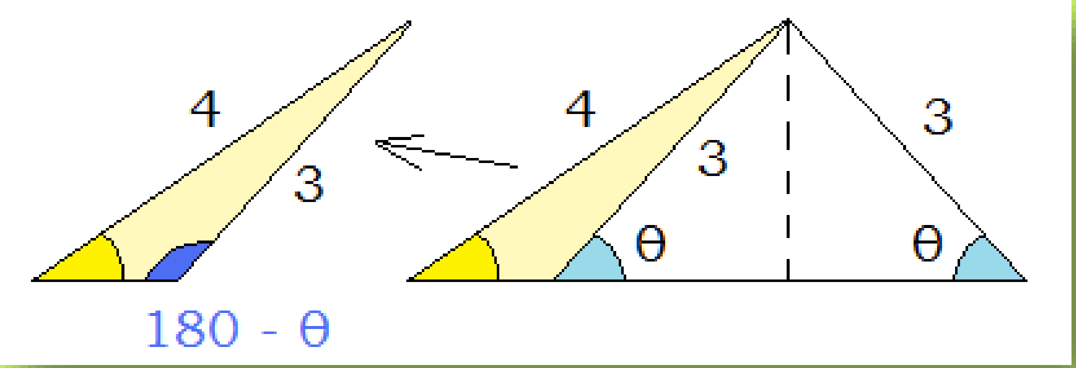
The Sine and Cosine Rules

Calculate the angle θ in the given triangle.



Solution: Coming up!

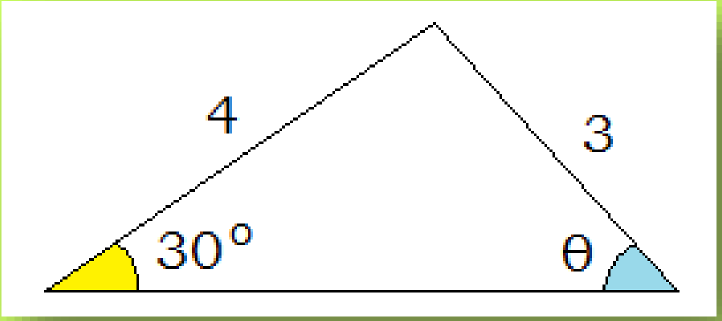
Since the side opposite the angle of 30° is less than 4 units it is possible for this side to tuck under the side of 4 units.



Consequently, there are two possible values for θ.
One answer is an acute angle θ_a (the principal value given by a calculator) and the other is an obtuse angle equal to $180 - \theta_a$.

The Sine and Cosine Rules

Solution: Continued ...



Apply the sine rule.

Since an angle is required write the sine rule with the sine functions as the numerators. (This saves working out!)

$$\frac{\sin \theta}{4} = \frac{\sin 30}{3}$$

Since sin 30 = 0.5, $\sin \theta = \frac{2}{3}$

Take the inverse sine, $\theta = \sin^{-1}(2/3) = 41.8^0$ (to 1 d.p.)

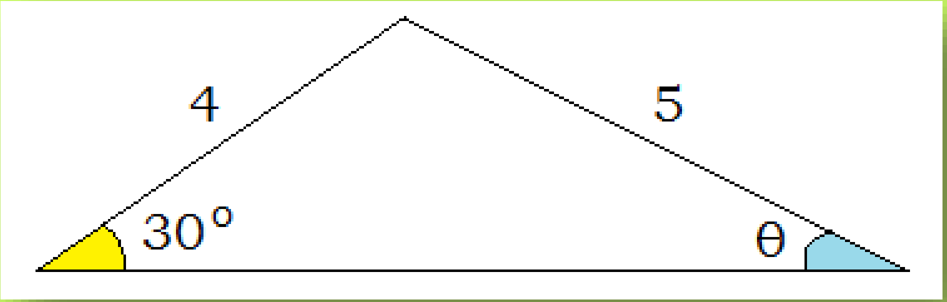
The obtuse angle for $\theta = 180^0 - 41.8^0 = 138.2^0$ (to 1 d.p.)

Therefore, $\theta = 41.8^0$ (to 1 d.p.) or $\theta = 138.2^0$ (to 1 d.p.)

The Sine and Cosine Rules

Example 15a(ii)

Calculate the angle θ in the given triangle.



Solution: Coming up!

This time the side opposite the angle of 30° is too long to tuck under the side of 4 units.

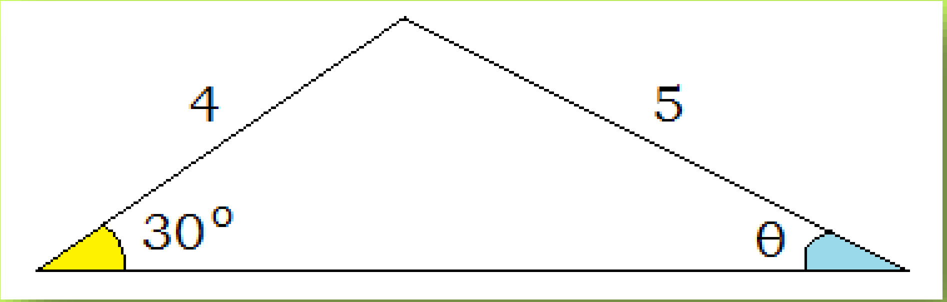
Consequently, there is only one possible θ value in this case.

Note: This shows that it is not a matter of finding the principal value on a calculator and then blindly giving (180 – principal value) as a second value for θ.

You have to analyse the given triangle to see how many possible answers there are.

The Sine and Cosine Rules

Solution: Continued ...



Apply the sine rule.

Since an angle is required write the sine rule with the sine functions as the numerators. (This saves working out!)

$$\frac{\sin\theta}{4} = \frac{\sin 30}{5}$$

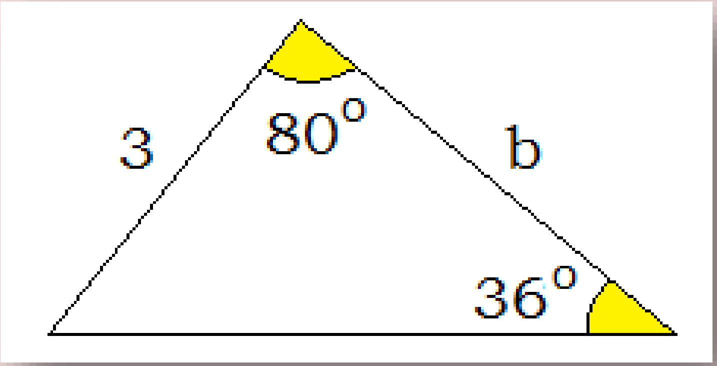
Since sin 30 = 0.5, $\sin\theta = \frac{2}{5}$

Take the inverse sine, $\theta = \sin^{-1}(2/5) = 23.6^0$ (to 1 d.p.)

Therefore, $\theta = 23.6^0$ (to 1 d.p.)

The Sine and Cosine Rules

Exercise 15a



Find the value of b. (Give the result to 3 significant figures.)

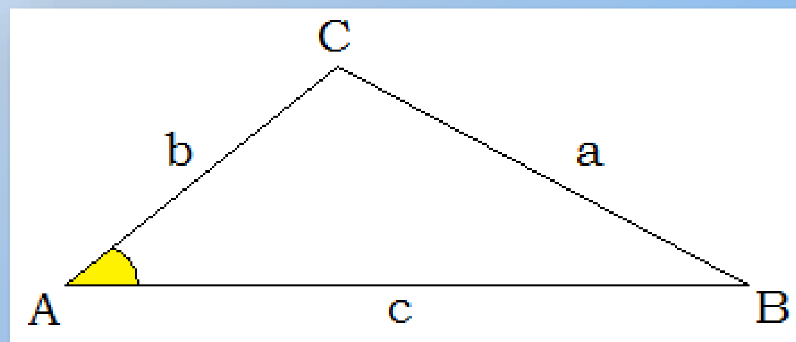
Write your result on the answer sheet.

The Sine and Cosine Rules

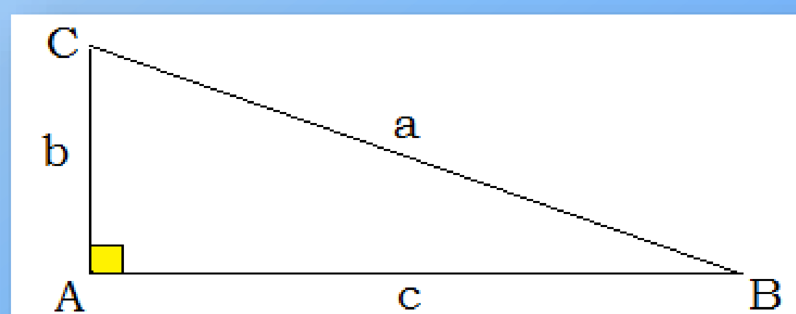
Derivation of the Cosine Rule

The cosine rule, applied to the triangle ABC, is:

$$a^2 = b^2 + c^2 - 2bc \cos A$$



Proof: Start with the case where angle A is a right-angle.



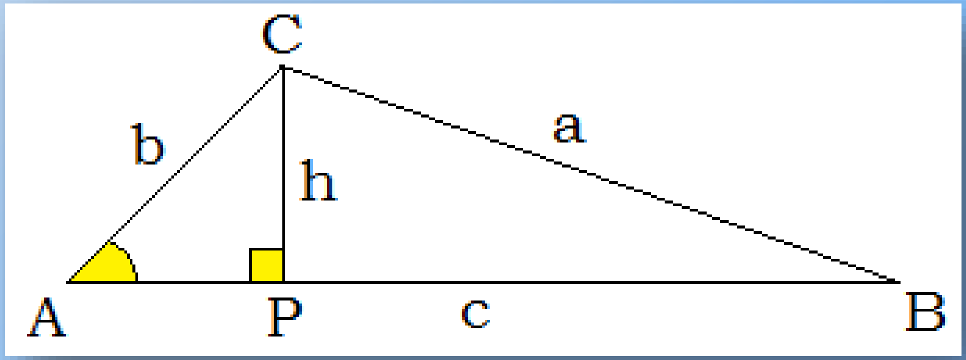
Pythagoras' theorem applies in this case: $a^2 = b^2 + c^2$

The Sine and Cosine Rules

Derivation of the Cosine Rule

Tilt the side AC so that the length of the side BC decreases compared to the case for a right-angled triangle (keeping b and c the same).

Drop a vertical line, of length h, from C to meet the line AB at point P.



From triangle APC, $h^2 = b^2 - (AP)^2$

From triangle CPB, $h^2 = a^2 - (c - AP)^2$

Equate h^2 to get: $a^2 - (c - AP)^2 = b^2 - (AP)^2$

The Sine and Cosine Rules

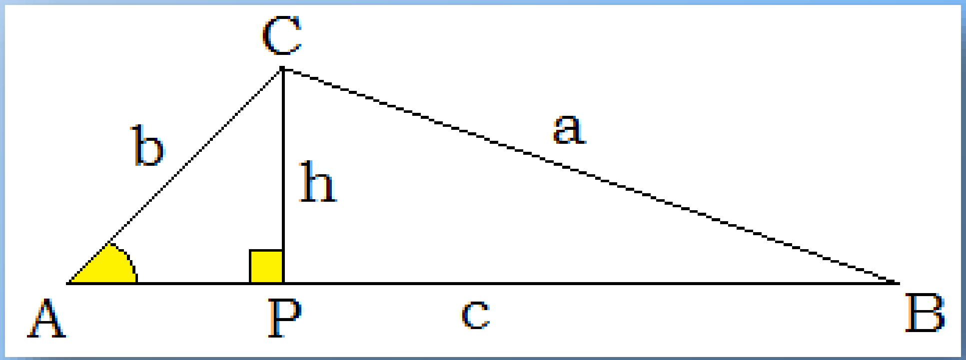
Derivation of the Cosine Rule

Recap: $a^2 - (c - AP)^2 = b^2 - (AP)^2$

$$a^2 - (c^2 - 2cAP + \{AP\}^2) = b^2 - (AP)^2$$

$$a^2 - c^2 + 2cAP - \{AP\}^2 = b^2 - (AP)^2$$

Rearrange to get: $a^2 = b^2 + c^2 - 2cAP$



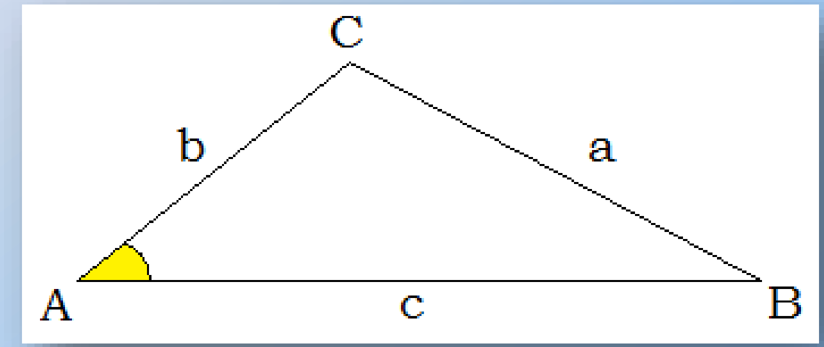
From triangle APC, $AP = b \cos A$

Therefore, $a^2 = b^2 + c^2 - 2bc\cos A$

This is the cosine rule.

The Sine and Cosine Rules

Derivation of the Cosine Rule



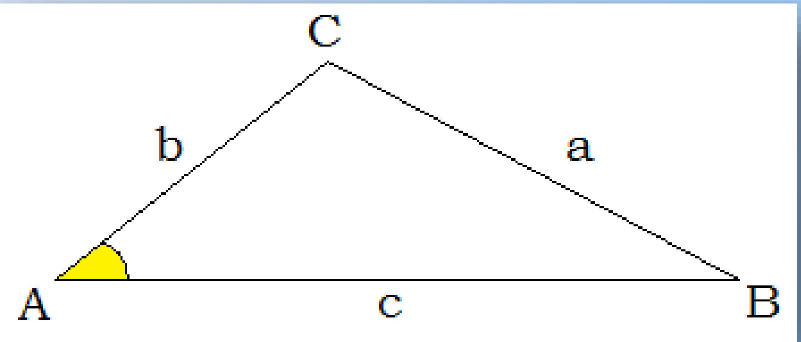
Note that the cosine rule is essentially Pythagoras' theorem with an extra term to account for the tilting angle A.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Since $\cos 90^o = 0$ the cosine rule reduces to Pythagoras' theorem when angle A is a right-angle.

The Sine and Cosine Rules

The cosine rule can be used to calculate an angle when the lengths of the three sides of the triangle are known.



Make angle A the subject of the cosine rule.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Rearrange to get: $2bc \cos A = b^2 + c^2 - a^2$

Divide by 2bc, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

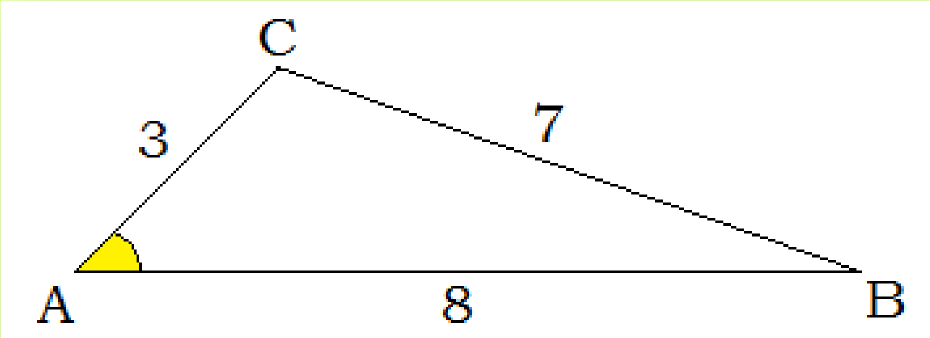
Take the inverse cosine to get:

$$\text{Angle } A = \cos^{-1}\left\{\frac{b^2 + c^2 - a^2}{2bc}\right\}$$

The Sine and Cosine Rules

Example 15b

Calculate the angle A.



Solution: Coming up!

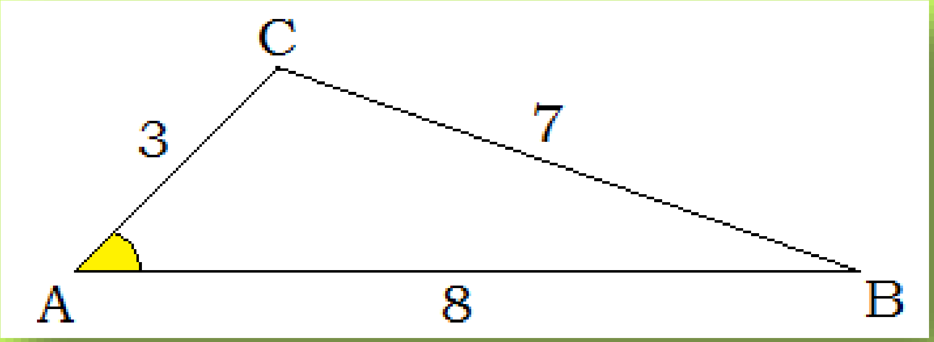
When working with the cosine rule it is easier to work with the structure rather than using the same letters all the time.

The numerator of the rearranged cosine rule has the squares of the sides and the negative term relates to the side opposite the required angle.

The denominator of the rearranged cosine rule is twice the product of the sides adjacent to the required angle.

The Sine and Cosine Rules

Calculate the angle A.



Apply the rearranged cosine rule.

$$\text{Angle } A = \cos^{-1}\left\{ \frac{3^2 + 8^2 - 7^2}{2\{3\}\{8\}} \right\}$$

$$= \cos^{-1}\left\{ \frac{24}{48} \right\}$$

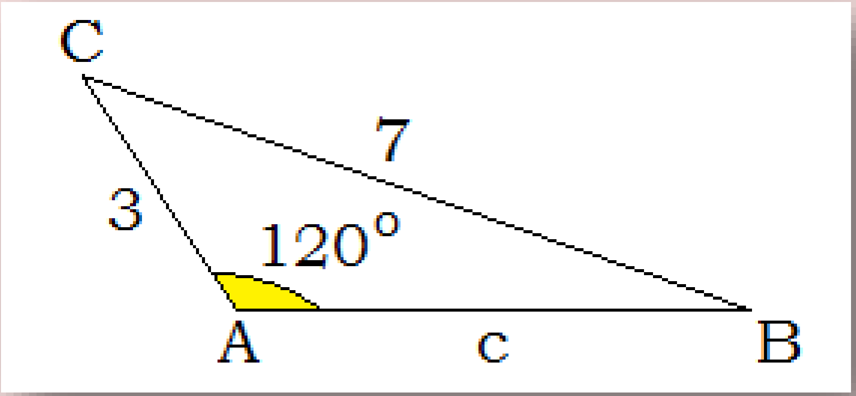
$$= \cos^{-1}\{ 0.5 \}$$

$$= 60^{o}$$

Therefore, the angle A = 60^{o} (exact)

The Sine and Cosine Rules

Exercise 15b



(a) Find a quadratic equation in c (the length of side AB).

(b) Solve this quadratic equation to find the value of c.

(c) Find a different set of natural numbers for the sides of a triangle which has an angle of 120°.

Write your results on the answer sheet.

Vulnavia

Ethereal Princess

Now provide a brief personal statement on the official answer sheet.

Please include:

your aspirations regarding mathematics and your favourite way to learn.

Send your completed answer sheet to:

Chancellor@nightparklane.com

Vulnavia

Ethereal Princess

The Road to Mora

Produced at Night Park Lane,

The Ethereal Academy of Mathematics

and Astronomy.

Chancellor@nightparklane.com