The r^{th} term of a Quadratic Sequence

A quadratic sequence is one in which the terms can be produced using a generating formula that is given by a quadratic function.

$$T_r = a r^2 + b r + c$$

Notation

T_r represents a term in the sequence,

where r represents the position of the term in the sequence.

For example, for the sequence: 1 4 9 16 25 ....

r = 1 r = 2 r = 3 r = 4 r = 5

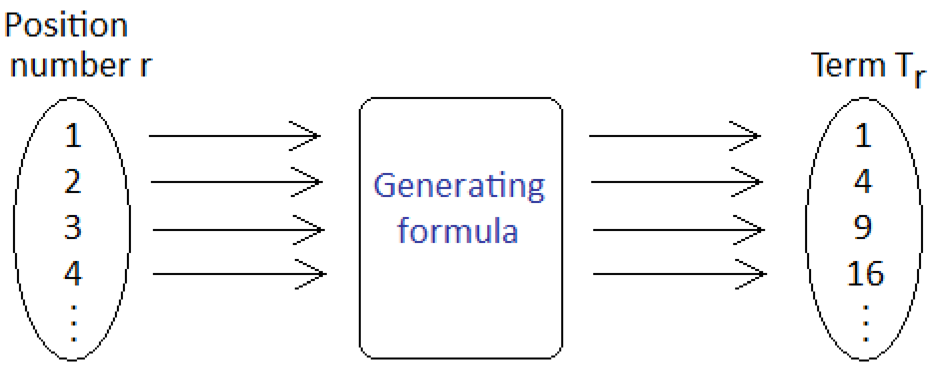
the first term T_1 is 1

the second term T_2 is 4

the third term T_3 is 9

... etc.

The aim is to produce a formula which gives the term T_r of the sequence for a given position number r.



You have to ask the question, "What do you have to do to the position number r, each time, to get the corresponding term in the sequence?"

In this case it can be seen by inspection that the sequence is just the square numbers sequence so the formula in this case is:

$$T_r = r \times r$$

That is, $T_r = r^2$

Once the generating formula has been found it can be used to find other terms in the sequence.

For example, $T_{10} = 10^2 = 100$.

Note that this formula is the general formula with $a = 1$, $b = 0$ and $c = 0$.

Before studying quadratic sequences in detail, a quick review of linear sequences follows.

The r^{th} term of a Quadratic Sequence

The Generating Formula of a Linear Sequence

A linear sequence is one in which the terms can be produced using a generating formula that is given by a linear function.

$L_r = mr + k$ (where m and k are constants).

A linear sequence has **equal gaps** between the terms.

For example, consider the numbers in the 5 times table.

5	10	15	20	25 ...	L_r ....
gap 5	gap 5	gap 5	gap 5		
r = 1	r = 2	r = 3	r = 4	r = 5	r = r

In this case the generating formula is: $L_r = 5r$

So the number in the formula in front of r (known as "the coefficient of r") is equal to the gap between the numbers.

Consider the general linear sequence where the first term is represented by p and the constant gap is d.

p	p + d	p + 2d	p + 3d	p + 4d ...	L_r: p + (r - 1) d ....
gap d	gap d	gap d	gap d		
r = 1	r = 2	r = 3	r = 4	r = 5	r = r

So the general term is given by the formula: $L_r = p + (r - 1)d$

[Note: The number in front of d is always 1 less than the position of the term in the sequence.]

This formula can be written as: $L_r = p + rd - d$

That is $L_r = dr + p - d$

Compare this with the formula: $L_r = mr + k$

So the number in front of r is equal to the gap (d) between the numbers and the constant term (k) in the formula is equal to (p – d).

For example, for the 5 times table sequence: the gap d = 5

and $p - d = 5 - 5 = 0$ So the r^{th} term is: $L_r = 5r$

Example: Find the generating formula of the sequence:

9 16 23 30 37 .....

The gap d between the numbers is: d = 7
The constant term k = p – d where, in this case, p = 9 and d = 7. So , p – d = 9 – 7 = 2.

So the general term (given by the generating formula) is: $L_r = 7r + 2$

[Note: Each term is given by the corresponding number in the seven times table **plus 2**.]

The r^{th} term of a Quadratic Sequence

Finding the Generating Formula of a Quadratic Sequence by Using the Method of Differences.

Start by considering the square numbers sequence:

1 4 9 16 25 ....

r = 1 r = 2 r = 3 r = 4 r = 5

Apply the method of differences.

Examine the gaps between the terms.

1 4 9 16 25 ... T_r ....

3 5 7 9

r = 1 r = 2 r = 3 r = 4 r = 5 r = r

Now produce the second row of differences.

1 4 9 16 25 ... T_r ....

3 5 7 9

2 2 2

Notice that the numbers in the second row of differences are all equal.

This is a crucial property of all quadratic sequences.

> The numbers in the second row of differences
> of any quadratic sequence
> are all equal.

We know that the formula for the square numbers sequence is:

$$T_r = r^2$$

How can this formula be found using the constant number in the second row of differences?

It turns out the number in front of r^2 (known as the coefficient of r^2) is always half the constant number in the second row of differences.

So if the constant number in the second row of differences is labelled as g

then the coefficient of r^2 in the generating formula: $T_r = a r^2 + b r + c$

is given by: $a = \frac{g}{2}$

Examine the terms in the sequence with the generating formula $\left(\frac{g}{2}\right) r^2$ in the case where $g = 2$.

The generating formula r^2 gives, 1 4 9 16 25 ...

This gives the required quadratic sequence directly.

Therefore, the general term of the given sequence is: $T_r = r^2$

The r^{th} term of a Quadratic Sequence

Example 1 Find the general term of the quadratic sequence:

2 15 34 59 90 127 ...

Solution

The general term has the form: $T_r = a r^2 + b r + c$

Find 'a', the coefficient of r^2.

Apply the method of differences to find the constant number (g) in the second row of differences.

2		15		34		59		90		127 ...
	13		19		25		31		37	
		6		6		6		6		

So, $g = 6$

Since the coefficient of r^2 in the generating formula of a quadratic sequence is always equal to half the value of g then: $a = 3$

Thus, the quadratic term in the generating formula is: $3r^2$

[Exam warning!

Examiners report ,that in questions such as this, many students would just leave the answer as $3r^2$.

But this is only the quadratic term in the required formula.

$$T_r = 3r^2 + b r + c$$

The generating formula for the linear sequence: $b r + c$ still has to be found.]

Each term in the given quadratic sequence

= the corresponding term in the quadratic sequence given by $3r^2$ + the corresponding term in the linear sequence given by $b r + c$

To find each term in the linear sequence, find out the number which has to be added to the sequence described by the formula $3r^2$ in order to obtain the corresponding term in the given quadratic sequence.

Present the sequences as follows.

	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6	
T_r	2	15	34	59	90	127	...
$3r^2$	3	12	27	48	75	108	...
Linear sequence	-1	3	7	11	15	19	...

Example 1 (Continued)

Examine the linear sequence: -1 3 7 11 15 19 ......

The generating formula for a linear sequence is: $L_r = d\,r + p - d$
where d is the constant gap and p is the first term.

Since p = -1 and d = 4 then the generating formula for the linear sequence is:

$$L_r = 4\,r + (-1 - 4)$$

That is, $L_r = 4\,r - 5$

Comparing this with $b\,r + c$ gives: $b = 4$ and $c = -5$

Now, the general term of the quadratic sequence T_r = $3r^2$ + the general term of the linear sequence L_r

Thus, $T_r = 3r^2 + 4\,r - 5$

Check: Test the formula to ensure that it gives the required terms in the given quadratic sequence.

Recall the given sequence: 2 15 34 59 90 127 ...

Test r = 1 in the formula.

$$T_1 = 3(1)^2 + 4(1) - 5 = 3 + 4 - 5 = 2 \text{ (as required)}$$

Test r = 5 in the formula.

$$T_5 = 3(5)^2 + 4(5) - 5 = 75 + 20 - 5 = 90 \text{ (as required)}$$

Therefore, the general term T_r of the given quadratic sequence is:

$$T_r = 3r^2 + 4r - 5$$

Exercise 1.1 Find the general term of the quadratic sequence:

-4 3 18 41 72 111 ...

Exercise 1.2 Find the general term of the quadratic sequence:

3 6 5 0 -9 -22 ...

Next, study the tutorial; Quadratic Sequence (Part 2)