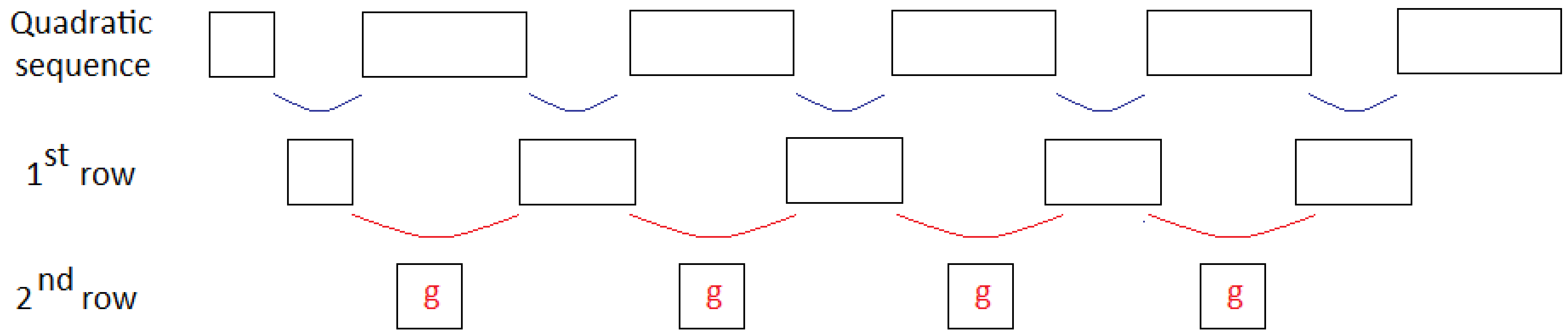
The r^{th} term of a Quadratic Sequence

Using MS Excel to Find the General Term of a Quadratic Sequence

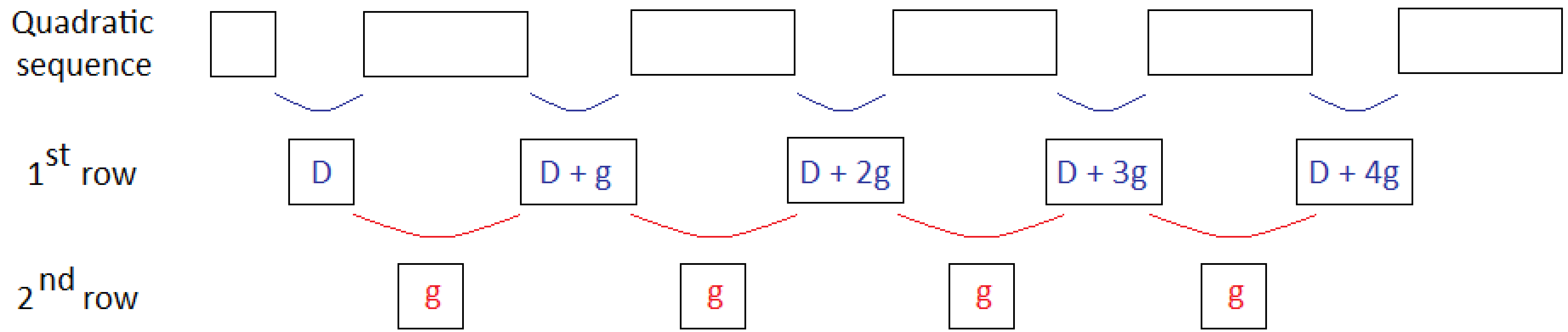
To produce an efficient method to find the general term of any given quadratic sequence, find the general solution using the method of differences.

Start with the second row of differences which has a constant number g.



Let D = the gap between the first two numbers in the quadratic sequence. [Capital D is used to distinguish it from d (the constant gap in the linear sequence).]

Then fill in the first row of differences.



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The r^{th} term of a Quadratic Sequence

Let P = the first term of the quadratic sequence. [Capital P is used to distinguish it from p (the first term of the general linear sequence).]

Quadratic sequence	P										
1^{st} row		D		D + g		D + 2g		D + 3g		D + 4g	
2^{nd} row			g		g		g		g		

The second term in the quadratic sequence = P + D

The third term in the quadratic sequence = P + D + (D + g) ≡ P + 2D + g

The fourth term in the quadratic sequence = P + D + (D + g) + (D + 2g) ≡ P + 3D + 3g

The fifth term in the quadratic sequence = P + D + (D + g) + (D + 2g) + (D + 3g) ≡ P + 4D + 6g

The sixth term in the quadratic sequence = P + D + (D + g) + (D + 2g) + (D + 3g) + (D + 4g) ≡ P + 5D + 10g

Quadratic sequence	P		P + D		P + 2D + g		P + 3D + 3g		P + 4D + 6g		P + 5D + 10g
1^{st} row		D		D + g		D + 2g		D + 3g		D + 4g	
2^{nd} row			g		g		g		g		

Since the constant number in the second row of differences is g then the quadratic term in the generating formula is: $\left(\frac{g}{2}\right) r^2$

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The r^{th} term of a Quadratic Sequence

Each term in the general quadratic sequence

= the corresponding term in the quadratic sequence given by $\left(\frac{g}{2}\right) r^2$ + the corresponding term in the linear sequence given by $b\,r + c$

To find each term in the linear sequence, find out the number which has to be added to the sequence described by the formula $\left(\frac{g}{2}\right) r^2$

in order to obtain the corresponding term in the general quadratic sequence.

	r = 1	r = 2	r = 3	r = 4	r = 5	r = 6
T_r	$\boxed{P}$	$\boxed{P + D}$	$\boxed{P+2D+g}$	$\boxed{P+3D+3g}$	$\boxed{P+4D+6g}$	$\boxed{P+5D+10g}$
$\frac{g}{2}\, r^2$	$\frac{g}{2}$	$2g$	$\frac{9}{2}g$	$8g$	$\frac{25}{2}g$	$18g$
Linear sequence	$P - \frac{g}{2}$	$P + D - 2g$	$P+2D - \frac{7}{2}g$	$P+3D - 5g$	$P+4D - \frac{13}{2}g$	$P+5D - 8g$

Examine the linear sequence.

Recall that the generating formula for a linear sequence is: $L_r = d\,r + p - d$
where d is the constant gap and p is the first term.

Note that the first term (p) of the linear sequence is $(P - \frac{g}{2})$. That is, $p = P - \frac{g}{2}$

The gap between the terms is constant. So this gap is given by the first term subtracted from the second term.

Thus, $d = P + D - 2g - (P - \frac{g}{2})$ This simplifies to become: $d = D - \frac{3}{2}g$

Note that $p - d = P - \frac{g}{2} - (D - \frac{3}{2}g)$ This simplifies to become: $p - d = P - D + g$

So the general term of this linear sequence is: $L_r = (D - \frac{3}{2}g)\,r + P - D + g$

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The r^{th} term of a Quadratic Sequence

Recall: The general term of this linear sequence is: $L_r = \left(D - \frac{3}{2}g\right) r + P - D + g$

Comparing this with $b\,r + c$ gives: $b = D - \frac{3}{2}g$ and $c = P - D + g$

Now, the general term of the quadratic sequence T_r $= \left(\frac{g}{2}\right) r^2 +$ the general term of the linear sequence L_r

Thus,

$$\boxed{T_r = \left(\frac{g}{2}\right) r^2 + \left(D - \frac{3}{2}g\right) r + P - D + g}$$

Although this formula gives the general term of the general quadratic sequence it uses the gap D between the first two numbers and the constant number g in the second row of differences.

Express the general term using **the first three terms** of the general quadratic sequence.

Write the general quadratic sequence as: P Q R .....

Since D is the gap between the first two terms, $D = Q - P$.

The first three terms of the general quadratic sequence were previously expressed as: $P \quad (P + D) \quad (P + 2D + g)$

Note that the third term – the first term

$= P + 2D + g - P$

$\equiv 2D + g$

But this must be equal to; $R - P$

Thus, $2D + g = R - P$

Rearrange to get: $g = R - P - 2D$

But $D = Q - P$, so; $g = R - P - 2(Q - P)$ Thus, $g = P - 2Q + R$

The r^{th} term of a Quadratic Sequence

Recap:

$$T_r = \left(\frac{g}{2}\right) r^2 + \left(D - \frac{3}{2}g\right) r + P - D + g$$

When the general quadratic sequence is written as: P Q R ....

it has been shown that: $D = Q - P$ and $g = P - 2Q + R$

So the coefficient of $r^2 = \frac{g}{2} \equiv \frac{1}{2}(P - 2Q + R)$

The coefficient of $r = D - \frac{3}{2}g \equiv Q - P - \frac{3}{2}(P - 2Q + R)$

$$\equiv -\frac{5}{2}P + 4Q - \frac{3}{2}R$$

$$\equiv \frac{1}{2}(-5P + 8Q - 3R)$$

The constant term $= P - D + g \equiv P - (Q - P) + (P - 2Q + R)$

$$\equiv 3P - 3Q + R$$

So the general term of the general quadratic sequence expressed using the first three terms P, Q and R is:

$$T_r = \frac{1}{2}(P - 2Q + R)\, r^2 + \frac{1}{2}(-5P + 8Q - 3R)\, r + 3P - 3Q + R$$

This is not the sort of formula to memorise for use in an exam!

However, it is very useful to use in a spreadsheet since it gives the general term without needing the method of differences.

See the MS Excel computer file: Quadratic Sequence Formula

The r^{th} term of a Quadratic Sequence

Example 1 Test the general generating formula for the sequence: 1 4 9 16 25 ....

Solution Since the first three terms of the given sequence are the first three square numbers, in order, then the general term must be: $T_r = r^2$.
So this result must be obtained by using the general formula.

For the sequence 1 4 9 .... set P = 1, Q = 4 and R = 9 in the general formula:

$$\boxed{T_r = \frac{1}{2}(P - 2Q + R)\, r^2 + \frac{1}{2}(-5P + 8Q - 3R)\, r + 3P - 3Q + R}$$

The coefficient of r^2 is: $\frac{1}{2}(P - 2Q + R) = \frac{1}{2}(1 - 2\{4\} + 9) = \frac{1}{2}(2) = 1$ as required.

The coefficient of r is: $\frac{1}{2}(-5P + 8Q - 3R) = \frac{1}{2}(-5\{1\} + 8\{4\} - 3\{9\}) = \frac{1}{2}(-5 + 32 - 27) = 0$ as required.

The constant term is: $3P - 3Q + R = 3\{1\} - 3\{4\} + 9 = 3 - 12 + 9 = 0$ as required.

Therefore, the generating formula is: $T_r = r^2$ (as anticipated!)

What happens when the first two terms of the sequence: 1 4 9 .... are switched around?

Consider the sequence: 4 1 9 ....

Although the numbers go down then up again, this does not cause a problem.

Note that when the terms of a linear sequence are plotted, a straight line is produced.
Once two points are given there is only one straight line which can be drawn through these points. In other words, the equation of the line is fixed.

When the terms of a quadratic sequence are plotted, a curve known as a parabola is produced.
Once three points are given there is only one parabola which can be drawn through these points. In other words, the equation of the parabola is fixed.

Since a parabola can always be drawn through any three given points (provided none of them are vertically above each other!) it only requires three terms to determine the general term of a quadratic sequence.

The r^{th} term of a Quadratic Sequence

Example 2 Find the general term for the sequence: 4 1 9 .....

Solution: For the sequence 4 1 9 .... set P = 4, Q = 1 and R = 9 in the general formula:

$$\boxed{T_r = \frac{1}{2}(P - 2Q + R)\, r^2 + \frac{1}{2}(-5P + 8Q - 3R)\, r + 3P - 3Q + R}$$

The coefficient of r^2 is: $\frac{1}{2}(P - 2Q + R) = \frac{1}{2}(4 - 2\{1\} + 9) = \frac{11}{2}$

The coefficient of r is: $\frac{1}{2}(-5P + 8Q - 3R) = \frac{1}{2}(-5\{4\} + 8\{1\} - 3\{9\}) = \frac{1}{2}(-20 + 8 - 27) = -\frac{39}{2}$

The constant term is: $3P - 3Q + R = 3\{4\} - 3\{1\} + 9 = 12 - 3 + 9 = 18$

So the generating formula is: $T_r = \frac{11}{2} r^2 - \frac{39}{2} r + 18$

This generating formula can be expressed as:

$$T_r = \frac{1}{2}(11r^2 - 39r + 36)$$

Graphical analysis follows ...

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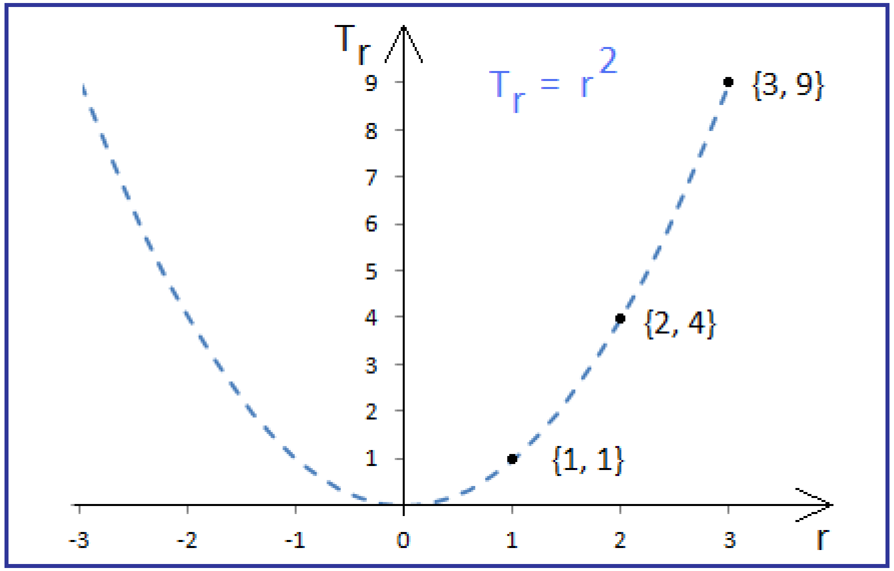
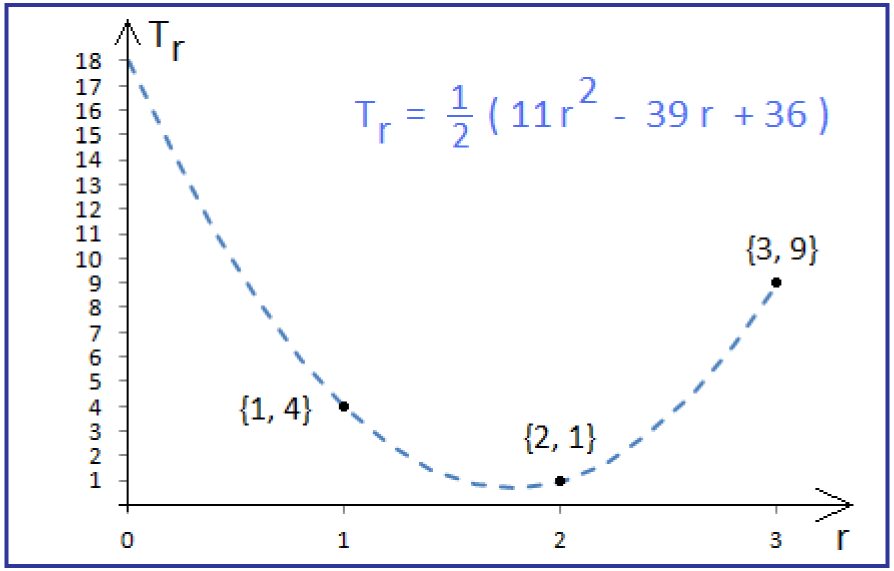
Graphical Analysis





This graph shows that the parabola with the equation:

$T_r = r^2$ passes through the points {1, 1}, {2, 4} and {3, 9}.

So, the general term of the sequence; 1 4 9 ....

is given by the formula $T_r = r^2$.

This graph shows that the parabola with the equation:

$T_r = \frac{1}{2}(11r^2 - 39r + 36)$ passes through the points {1, 4}, {2, 1} and {3, 9}.

So, the general term of the sequence; 4 1 9 ....

is given by the formula $T_r = \frac{1}{2}(11r^2 - 39r + 36)$.

Now try the exercises for Part 2.