The Art of Equations 1

by Anthony van Bïck

This art display examines an elegant way to solve simultaneous equations.

Start with two general equations in two unknowns x and y.

$$ax + cy = u \qquad \text{[Equation 1]}$$

$$dx + by = v \qquad \text{[Equation 2]}$$

This system of equations may be expressed in matrix form as follows.

$$\begin{pmatrix} a & c \\ d & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

The determinant of the coefficient matrix **C** gives the denominator of x and y.

$$|\mathbf{C}| \equiv \begin{vmatrix} a & c \\ d & b \end{vmatrix} = ab - cd$$

This determinant gives information about the type of solution.

The equations may be regarded as representing the equations of straight lines.

If ab = cd it can be shown that the lines have equal gradient and are therefore parallel. In such cases the system of equations cannot have a unique solution.

The Art of Equations 2

Now for the clever bit.

Define the augmented matrix for the given system of two equations as:

$$\left(\begin{array}{cc|c} a & c & u \\ d & b & v \end{array}\right)$$

Exclude x from this matrix. (That is, exclude the first column).

Define the determinant of this augmented matrix with x excluded as:

$$\left|\mathbf{A}_{aug}\right|_{\text{no } x} \equiv \begin{vmatrix} c & u \\ b & v \end{vmatrix} = cv - bu$$

Similarly, define the determinant of the augmented matrix with y excluded as:

$$\left|\mathbf{A}_{aug}\right|_{\text{no } y} \equiv \begin{vmatrix} a & u \\ d & v \end{vmatrix} = av - du$$

It can be shown, using algebra, that $x = -\left(\dfrac{cv - bu}{ab - cd}\right)$ and $y = \dfrac{av - du}{ab - cd}$

Thus

$$\boxed{x = -\frac{\left|\mathbf{A}_{aug}\right|_{\text{no } x}}{|\mathbf{C}|} \quad \text{and} \quad y = \frac{\left|\mathbf{A}_{aug}\right|_{\text{no } y}}{|\mathbf{C}|}}$$

Calculating x and y using these formulae avoids using algebra!

The Art of Equations 3

Now consider the geometry of the solution of the simultaneous equations.

$$x = -\frac{|A_{aug}|_{\text{no } x}}{|C|} \quad \text{and} \quad y = \frac{|A_{aug}|_{\text{no } y}}{|C|}$$

Case 1

If the determinant of the coefficient matrix **C** is nonzero then the system of equations will have a unique solution.

This is equivalent to two lines crossing at a point.

Case 2

If the determinant of the coefficient matrix **C** is zero then both x and y will have zero in the denominator. (Bad news mathematically!)

If $|A_{aug}|_{\text{no } x} = 0$ then x would be $\frac{0}{0}$ which is indeterminate which indicates that the system of equations has an infinite number of solutions.

This is equivalent to the two lines being superimposed so that each and every point on one line is automatically on the second line as well.

Case 3

Given that $|C| = 0$, if $|A_{aug}|_{\text{no } x} \neq 0$ then the magnitude of x would be infinite and the system of equations would have no solution.

This is equivalent to the situation where the two lines are parallel but separate so that there is no point which is on both lines.

The Art of Equations 4

The beauty of the determinant method is that a seamless transition can be made to solving a system of three simultaneous equations in three unknowns (x, y and z).

$$ax + dy + ez = u \qquad \text{[Equation 1]}$$

$$fx + by + gz = v \qquad \text{[Equation 2]}$$

$$hx + iy + cz = w \qquad \text{[Equation 3]}$$

It can be shown, using algebra, that:

$$x = \frac{|\mathbf{A}_{aug}|_{\text{no } x}}{|\mathbf{C}|}, \quad y = -\frac{|\mathbf{A}_{aug}|_{\text{no } y}}{|\mathbf{C}|} \quad \text{and} \quad z = \frac{|\mathbf{A}_{aug}|_{\text{no } z}}{|\mathbf{C}|}$$

The formulae may be written out in full as follows.

$$x = \frac{\begin{vmatrix} a & d & e & u \\ f & b & g & v \\ h & i & c & w \end{vmatrix}}{\begin{vmatrix} a & d & e \\ f & b & g \\ h & i & c \end{vmatrix}}, \quad y = \frac{-\begin{vmatrix} a & d & e & u \\ f & b & g & v \\ h & i & c & w \end{vmatrix}}{\begin{vmatrix} a & d & e \\ f & b & g \\ h & i & c \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a & d & e & u \\ f & b & g & v \\ h & i & c & w \end{vmatrix}}{\begin{vmatrix} a & d & e \\ f & b & g \\ h & i & c \end{vmatrix}}$$

The Art of Equations 5

The calculation of a 3 x 3 determinant.

$$\begin{vmatrix} a & d & e \\ f & b & g \\ h & i & c \end{vmatrix} = a \begin{vmatrix} b & g \\ i & c \end{vmatrix} - d \begin{vmatrix} f & g \\ h & c \end{vmatrix} + e \begin{vmatrix} f & b \\ h & i \end{vmatrix}$$

$$= a\,(bc - gi) - d\,(fc - gh) + e\,(fi - bh)$$

Exercise 1: Solve the system of equations:

$$\begin{aligned} 2x - 3y - 4z &= 7 \\ 3x + 5y + 8z &= 5 \\ 7x + 4y - 3z &= -15 \end{aligned}$$

The Art of Equations 6

Now consider the geometry of the solution of three simultaneous equations.

$$x = \frac{\left|\mathbf{A}_{aug}\right|_{no\ x}}{\left|\mathbf{C}\right|}, \quad y = -\frac{\left|\mathbf{A}_{aug}\right|_{no\ y}}{\left|\mathbf{C}\right|} \quad \text{and} \quad z = \frac{\left|\mathbf{A}_{aug}\right|_{no\ z}}{\left|\mathbf{C}\right|}$$

This time three planes are used instead of two lines.

Case 1

If the determinant of the coefficient matrix **C** is nonzero then the system of equations will have a unique solution.

This is equivalent to the three planes meeting at a point.

Case 2

If the determinant of the coefficient matrix **C** is zero then x, y and z will have zero in the denominator.

If $\left|\mathbf{A}_{aug}\right|_{no\ x} = 0$ then x would be $\frac{0}{0}$ which is indeterminate which indicates that the system of equations has an infinite number of solutions.

In this case the line of intersection of two of the planes is contained in the third plane. So there is an infinite number of points in all three planes.

Case 3

Given that $\left|\mathbf{C}\right| = 0$, if $\left|\mathbf{A}_{aug}\right|_{no\ x} \neq 0$ then the magnitude of x would be infinite and the system of equations would have no solution.

In this situation the planes form a triangular prism.
Since no point can be on all three faces of the prism the system of equations has no solution.

The Art of Equations 7

Now consider a system of 4 simultaneous equations in 4 unknowns (x, y, z, t).

The unique solution in this case is given by:

$$x = \frac{-\left|\mathbf{A}_{aug}\right|_{\text{no } x}}{|\mathbf{C}|}, \quad y = \frac{\left|\mathbf{A}_{aug}\right|_{\text{no } y}}{|\mathbf{C}|}, \quad z = \frac{-\left|\mathbf{A}_{aug}\right|_{\text{no } z}}{|\mathbf{C}|}, \quad t = \frac{\left|\mathbf{A}_{aug}\right|_{\text{no } t}}{|\mathbf{C}|}$$

A way to visualise this system is to consider four intersecting cubes.

If three of the cubes have a common line of intersection L, a unique solution will arise when a vertex of the fourth cube touches line L.

Things to ponder!

What is the physical meaning of t in the solution {x, y, z, t} given that there are only three spatial dimensions?

How can a system of five equations be represented geometrically?

The Art of Equations 8

Each item from $|\mathbf{C}|$, $|A_{aug}|_{\text{no t}}$, $|A_{aug}|_{\text{no z}}$, $|A_{aug}|_{\text{no y}}$, $|A_{aug}|_{\text{no x}}$ has a 4 x 4 determinant.
But each of these determinants can be expressed as a sum of four 3 x 3 determinants.

So the whole system requires a total of twenty 3 x 3 determinants to be solved.
However, it turns out that by symmetry there are ten pairs of 3 x 3 determinants.

The calculations can be tabulated as follows with the 3 x 3 determinants as coefficients of the elements of the first row of the augmented matrix. [$\mathbf{D}|_{\cancel{x},\cancel{k}}$ is a 3 x 3 determinant without x or k.]

	a_{11}	a_{12}	a_{13}	a_{14}	k_1
$\lvert\mathbf{C}\rvert$	$\mathbf{D}\big\rvert_{\cancel{x},\cancel{k}}$ 	$-\mathbf{D}\big\rvert_{\cancel{y},\cancel{k}}$ 	$\mathbf{D}\big\rvert_{\cancel{z},\cancel{k}}$ 	$-\mathbf{D}\big\rvert_{\cancel{t},\cancel{k}}$ 	
$\lvert\mathbf{A}_{aug}\rvert_{\text{no t}}$	$\mathbf{D}\big\rvert_{\cancel{x},\cancel{t}}$ 	$-\mathbf{D}\big\rvert_{\cancel{y},\cancel{t}}$ 	$\mathbf{D}\big\rvert_{\cancel{z},\cancel{t}}$ 		$-\mathbf{D}\big\rvert_{\cancel{t},\cancel{k}}$ 
$\lvert\mathbf{A}_{aug}\rvert_{\text{no z}}$	$\mathbf{D}\big\rvert_{\cancel{x},\cancel{z}}$ 	$-\mathbf{D}\big\rvert_{\cancel{y},\cancel{z}}$ 		$\mathbf{D}\big\rvert_{\cancel{z},\cancel{t}}$ 	$-\mathbf{D}\big\rvert_{\cancel{z},\cancel{k}}$ 
$\lvert\mathbf{A}_{aug}\rvert_{\text{no y}}$	$\mathbf{D}\big\rvert_{\cancel{x},\cancel{y}}$ 		$-\mathbf{D}\big\rvert_{\cancel{y},\cancel{z}}$ 	$\mathbf{D}\big\rvert_{\cancel{y},\cancel{t}}$ 	$-\mathbf{D}\big\rvert_{\cancel{y},\cancel{k}}$ 
$\lvert\mathbf{A}_{aug}\rvert_{\text{no x}}$		$\mathbf{D}\big\rvert_{\cancel{x},\cancel{y}}$ 	$-\mathbf{D}\big\rvert_{\cancel{x},\cancel{z}}$ 	$\mathbf{D}\big\rvert_{\cancel{x},\cancel{t}}$ 	$-\mathbf{D}\big\rvert_{\cancel{x},\cancel{k}}$